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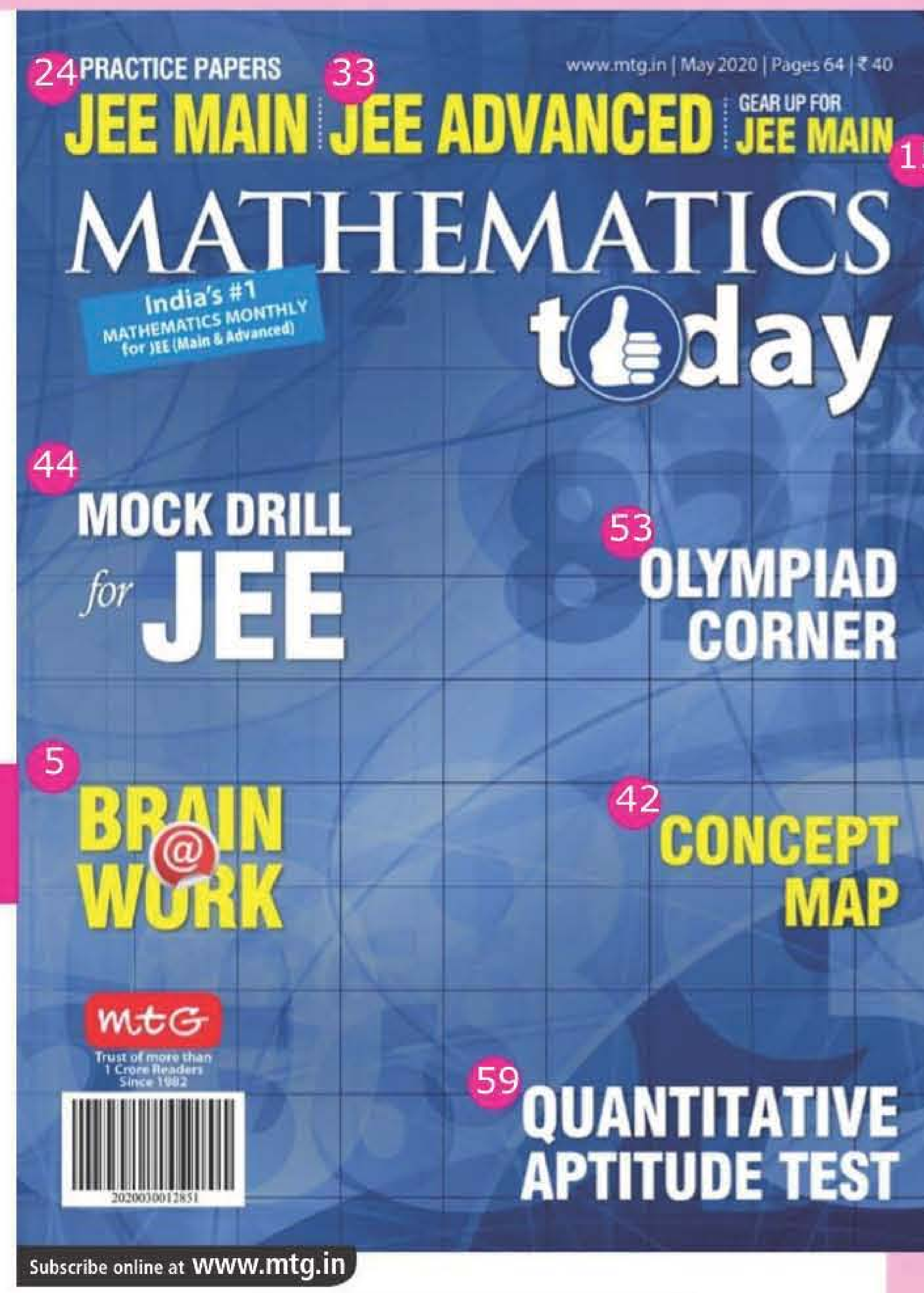

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BRAIN @ WORK

QUADRATIC EQUATIONS

IDENTITY

An identity is the statement of equality between two expressions which is always true for all values of the variables involved. So, $f(x) = g(x)$ is an identity if $f(x)$ and $g(x)$ have same value for every value of x.

Note:

(i) A polynomial of degree n represents an identity, if it is satisfied by $(n + 1)$ or more values of x.

(ii) If $f(x) = g(x)$ represents an identity, then the coefficients of similar terms of x are equal.

(iii) An equation $ax^3 + bx^2 + cx + d = 0$ represents an identity in terms of x, then $a = b = c = d = 0$.

PROPERTIES OF ROOTS OF EQUATION

Factor theorem

(i) $(x - \alpha)$ is a factor of a polynomial $f(x)$ if and only if $f(\alpha) = 0$.

(ii) $(x - \alpha)^2$ is a factor of a polynomial $f(x)$ if and only if $f(\alpha) = f'(\alpha) = 0$. In this case, we say that α is a repeated root of $f(x) = 0$ (a double root).

(iii) If $(x - \alpha)^m$ is a factor of a polynomial $f(x) = 0$, then $f(\alpha) = f'(\alpha) = f''(\alpha) = f'''(\alpha) = \ldots\ldots = f^{(m-1)}(\alpha) = 0$ and $f^m(\alpha) \neq 0$.

Remainder theorem

The value of remainder, when $f(x)$ is divided by $(x - \alpha)(x - \beta)$, is

$$\left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right)x+\left(\frac{af(\beta)-\beta f(\alpha)}{\alpha-\beta}\right).$$

Position of roots of a polynomial equation

If $f(x) = 0$ is an equation and a, b are two real numbers such that

(i) $f(a)\, f(b) < 0$, then the equation $f(x) = 0$ has at least one real root or an odd number of real roots between a and b.

(ii) If $f(a)$ and $f(b)$ are of the same sign, then either no real root or an even number of real roots of $f(x) = 0$ lie between a and b.

RELATION BETWEEN ROOTS AND CO-EFFICIENTS OF POLYNOMIAL EQUATION

Consider the general equation of n^{th} degree
$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots.. + a_{n-1} x + a_n = 0$, where $a_0, a_1, \ldots., a_n \in C$ and $n \in Z$
Let its roots be $\alpha_1, \alpha_2 \ldots., \alpha_n$. Then,

- Sum of roots taken one at a time $= S_1 = \sum \alpha_i = -\dfrac{a_1}{a_0}$

  Sum of product of roots taken two at a time $= S_2$
  $$= \sum_{i\neq j} \alpha_i\alpha_j = \frac{a_2}{a_0}$$
  Sum of product of roots taken three at a time $= S_3$
  $$= \sum_{i\neq j\neq k} \alpha_i\alpha_j\alpha_k = -\frac{a_3}{a_0}$$
  ....

  ....

  S_n = the product of roots taken all at a time $= S_n$
  $$= \alpha_1\alpha_2 \ldots.\, \alpha_n = (-1)^n \cdot \frac{a_n}{a_0}$$
  Number of terms in $S_1, S_2, S_3, \ldots., S_n$ are respectively ${}^nC_1, {}^nC_2, {}^nC_3, \ldots., {}^nC_n$.
- $f(\alpha_1) = f(\alpha_2) = f(\alpha_3) = f(\alpha_4) = \ldots. = f(\alpha_n) = 0$
- If $f(x) = 0$ has n real roots, then $f'(x) = 0$ has $(n - 1)$ real roots.
- If $f(x) = 0$ has n real roots, then $f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots. (x - \alpha_{n-1})(x - \alpha_n)$
- If $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots, \alpha_n$ are n roots of an equation, then the equation can be written as
  $x^n - S_1 x^{n-1} + S_2 x^{n-2} + \ldots + (-1)^n S_n = 0.$

TRANSFORMATION OF EQUATIONS

- An equation whose roots are reciprocals of the roots of a given equation is obtained by replacing x by $1/x$ in the given equation and simplify it to make it a polynomial equation.
- An equation whose roots are negative of the roots of a given equation is obtained by replacing x by $-x$ in the given equation and simplify it to make it a polynomial equation.
- An equation whose roots are squares of the roots of a given equation is obtained by replacing x by $\sqrt{x}$ in the given equation and simplify it to make it a polynomial equation.
- An equation whose roots are cubes of the roots of a given equation is obtained by replacing x by $x^{1/3}$ in the given equation and simplify it to make it a polynomial equation.

QUADRATIC EQUATION

An equation that can be written in the form $ax^2 + bx + c = 0\ \forall\ a, b, c \in R$ and $a \neq 0$, is called a quadratic equation.

The solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

The quantity $b^2 - 4ac$ is called the discriminant of the quadratic equation and is denoted by D or Δ.

Usually, the two roots of $ax^2 + bx + c = 0$ are denoted by α and β. The expression $ax^2 + bx + c$ can thus be written as $ax^2 + bx + c = a(x - \alpha)(x - \beta)$.

Sum and Product of Roots

$$\text{Sum of roots} = S = \alpha + \beta = \frac{-b}{a} = \frac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$$

$$\text{Product of roots} = P = \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Also, $(\text{Difference of roots})^2 = (\alpha - \beta)^2$

$$= (\alpha + \beta)^2 - 4\alpha\beta = \frac{b^2 - 4ac}{a^2} \Rightarrow |\alpha - \beta| = \frac{\sqrt{D}}{|a|}$$

The quadratic equation with sum of roots S and product of roots P is given by $x^2 - Sx + P = 0$

i.e., $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

The nature of the roots of $ax^2 + bx + c = 0$

(i) The roots are real and distinct iff $D > 0$.

(ii) The roots are real and equal iff $D = 0$ and the equal root is given by $x = -b/2a$.

When $D = 0$, $ax^2 + bx + c$ is a perfect square. Under this condition $ax^2 + bx + c = \left\{\sqrt{a}\left(x + \frac{b}{2a}\right)\right\}^2$

(iii) The roots are complex with non-zero imaginary part iff $D < 0$.

(iv) The roots are rational iff a, b, c are rational and D is a perfect square.

(v) The roots are of the form $p + \sqrt{q}$ $(p, q \in Q)$ *i.e.*, irrational iff a, b, c are rational and D is not a perfect square.

(vi) If a quadratic equation in x has more than two roots, then it is an identity in x.

Nature of the Roots of $f(x)\cdot g(x) = 0$

If D_1 and D_2 are the discriminants of the quadratic equations $f(x) = 0$ and $g(x) = 0$, then the following possibilities arises about the roots of the equation $f(x)\cdot g(x) = 0$ are

(i) If $D_1 + D_2 \geq 0$, then there will be at least two real roots of the equation $f(x) \cdot g(x) = 0$.

(ii) If $D_1 + D_2 < 0$, then there will be at least two imaginary roots of $f(x) \cdot g(x) = 0$.

(iii) If $D_1 \cdot D_2 < 0$, then the equation $f(x)\cdot g(x) = 0$ will have two real roots.

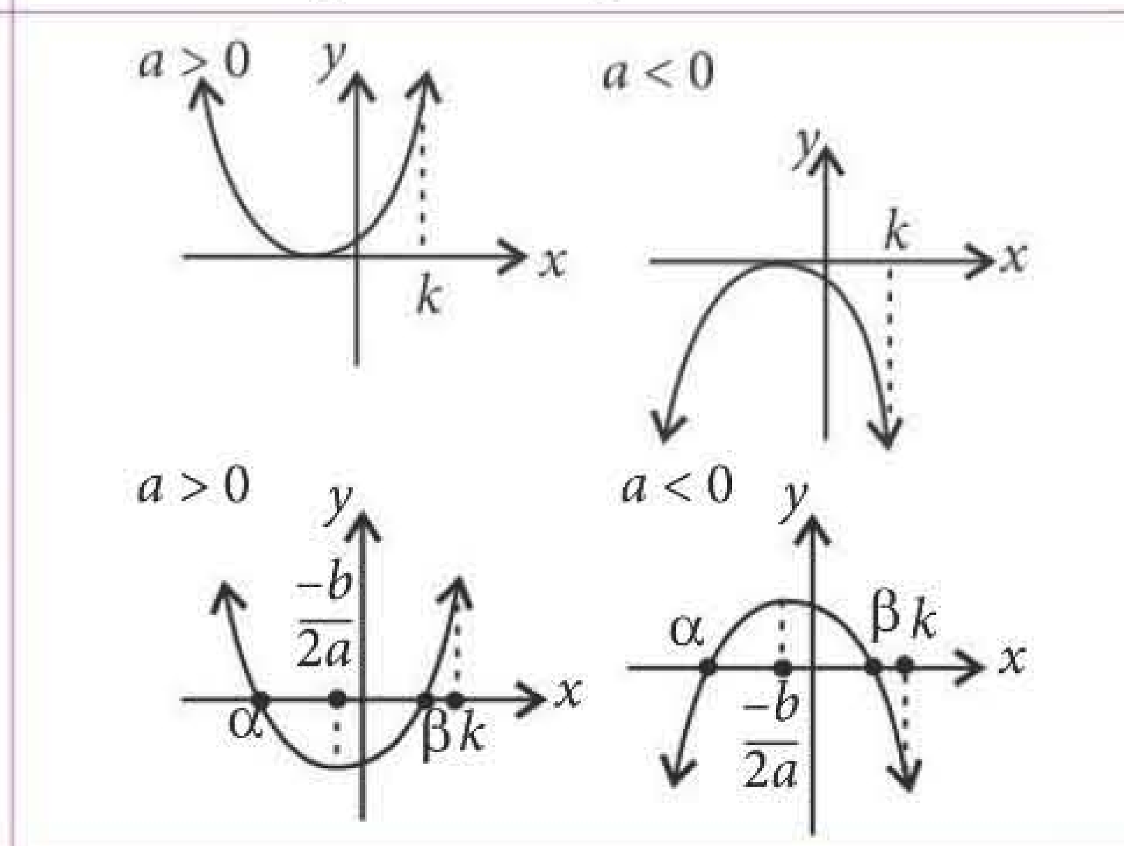
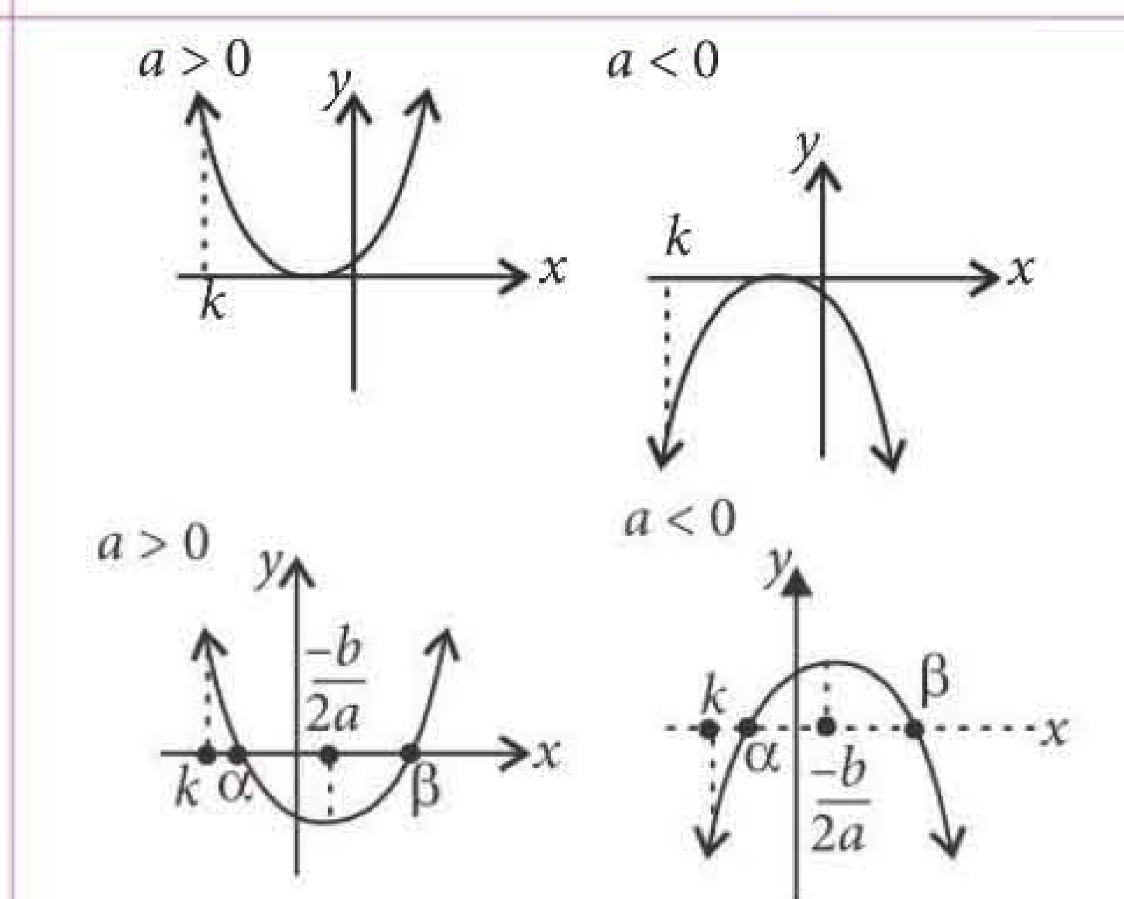
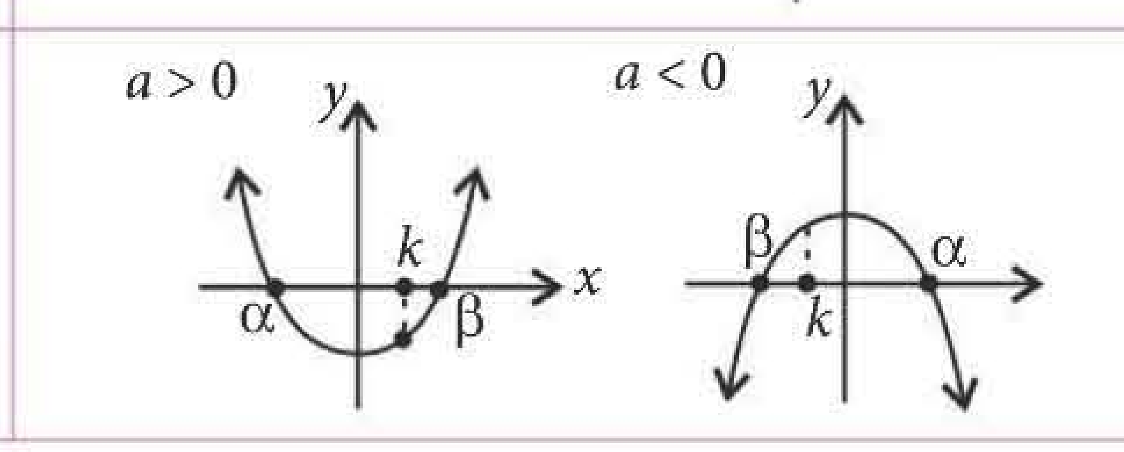
(iv) If $D_1 D_2 > 0$ then the equation $f(x)\cdot g(x) = 0$ has either four real roots or no real root.

CONDITION FOR COMMON ROOTS

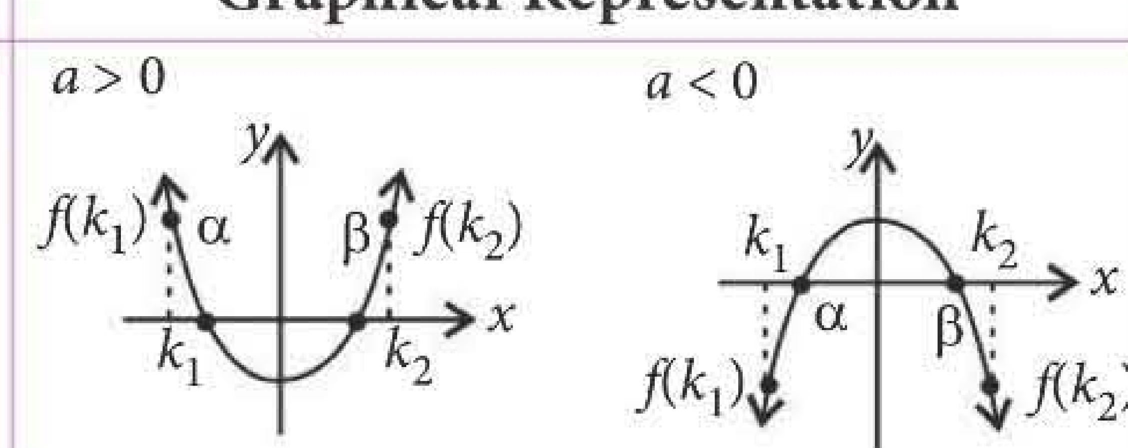
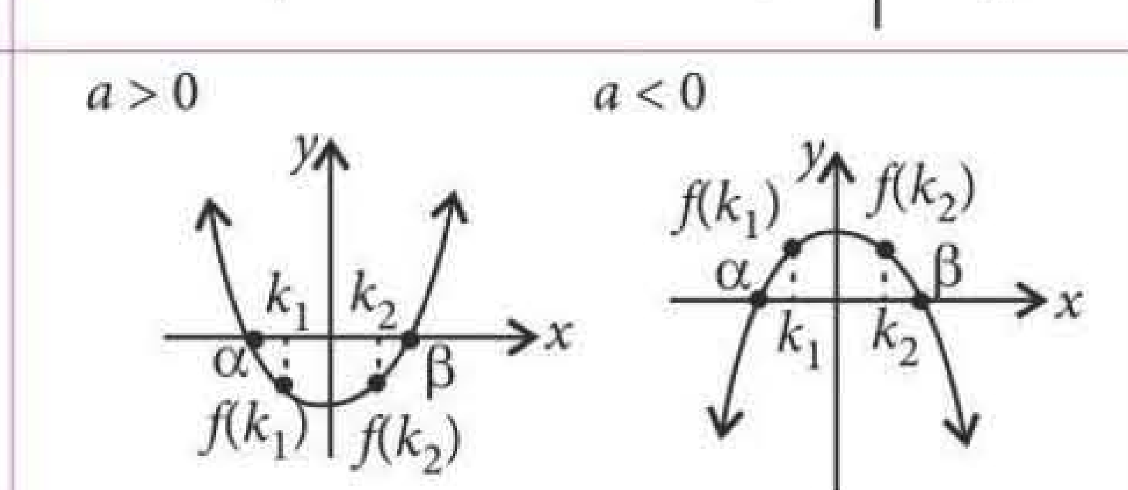
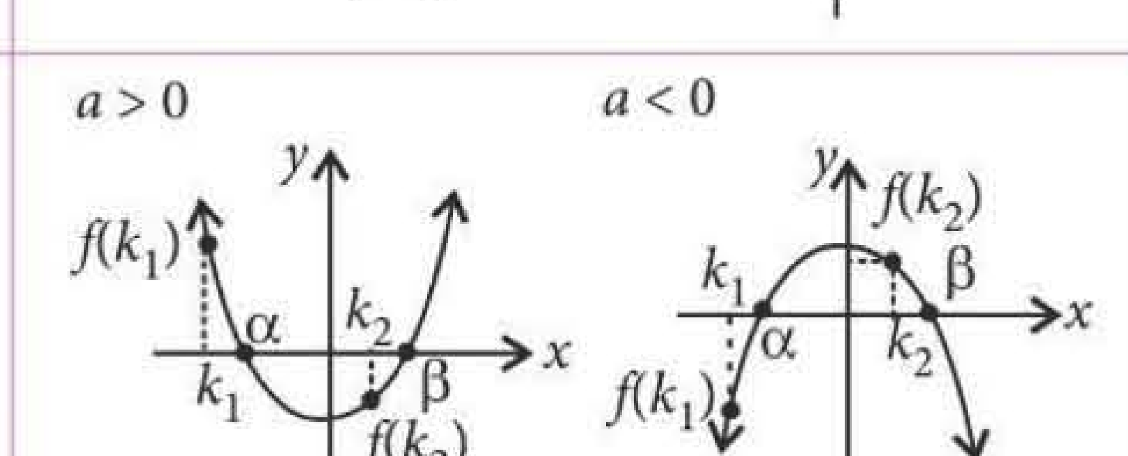
- If $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have one common root, then $(a_1b_2 - a_2b_1)(b_1c_2 - c_1b_2) = (c_1a_2 - a_1c_2)^2$
- If $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have both roots common, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- If the two equation $a_1x^2 + b_1x + c_1 = 0$, $a_2x^2 + b_2x + c_2 = 0$ with real coefficients have an imaginary root common, then both roots will be common, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- If the two equation $a_1x^2 + b_1x + c_1 = 0$; $a_2x^2 + b_2x + c_2 = 0$ with rational coefficients have an irrational root common, then both roots will be common, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- If every pair of three quadratic equations have a common root, then roots are taken as $\alpha, \beta; \beta, \gamma; \gamma, \alpha$
- If a quadratic equation and cubic equation have a common root, try to find the root of cubic equation by factorization.

POSITION OF ROOTS OF THE QUADRATIC EQUATION $ax^2 + bx + c = 0$

With Respect to One Quantity (k)

S.No.	Situation	Graphical Representation	Required Conditions
1.	Both the roots are less than k *i.e.*, $\alpha < \beta < k$		(i) $D \geq 0$ (ii) $a\,f(k) > 0$ (iii) $k > \frac{-b}{2a}$
2.	Both the roots are greater than k *i.e.*, $k < \alpha < \beta$		(i) $D \geq 0$ (ii) $a\,f(k) > 0$ (iii) $k < \frac{-b}{2a}$
3.	k lies between the roots *i.e.*, $\alpha < k < \beta$		(i) $D > 0$ (ii) $af(k) < 0$

With Respect to Two Quantities k_1 and k_2

S.No.	Situation	Graphical Representation	Required Conditions
1.	Distinct roots lies in the interval (k_1, k_2) *i.e.*, $k_1 < \alpha < \beta < k_2$		(i) $D > 0$ (ii) $a\,f(k_1) > 0$ (iii) $af(k_2) > 0$ (iv) $k_1 < \frac{-b}{2a} < k_2$
2.	Interval (k_1, k_2) lies between the roots *i.e.*, $\alpha < k_1 < k_2 < \beta$		(i) $D > 0$ (ii) $a\,f(k_1) < 0$ (iii) $af(k_2) < 0$
3.	One root lies in the interval (k_1, k_2) *i.e.*, $k_1 < \alpha < k_2 < \beta$		(i) $D > 0$ (ii) $f(k_1)\,f(k_2) < 0$

PROBLEMS

1. The natural number n for which the expression $y = 5(\log_3 n)^2 - \log_3 n^{12} + 9$, has the minimum value is
(a) 2 (b) 3 (c) $3^{6/5}$ (d) 4

2. If α, β are the roots of the quadratic equation $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + px - r = 0$, then $(\alpha - \gamma)(\alpha - \delta)$ is equal to
(a) $q + r$ (b) $q - r$
(c) $-(q + r)$ (d) $-(p + q + r)$

3. If α and β be the roots of the equation $x^2 + 3x + 1 = 0$, then the value of $\left(\frac{\alpha}{1+\beta}\right)^2 + \left(\frac{\beta}{\alpha+1}\right)^2$ is equal to
(a) 15 (b) 18 (c) 21 (d) none

4. The roots of the equation $x^2 + 6x + a = 0$ are real and distinct and they differ by atmost 4, then the range of value of a, is
(a) $(5, 9]$ (b) $[5, 9)$ (c) $[4, 8)$ (d) $[3, 9)$

5. If the equation $\cot^4 x - 2\,\text{cosec}^2 x + a^2 = 0$ has atleast one solution then, sum of all possible integral values of 'a' is equal to
(a) 4 (b) 3 (c) 2 (d) 0

6. If the equation $4x^2 - 4(5x + 1) + p^2 = 0$ has one root equals to two more than the other, then the value of p is equal to
(a) $\pm\frac{\sqrt{236}}{3}$ (b) ± 5
(c) 5 or -1 (d) 4 or -3

7. The values of k for which the quadratic equation $(1 - 2k)x^2 - 6kx - 1 = 0$ and $kx^2 - x + 1 = 0$ have atleast one root in common are
(a) $\left\{\frac{1}{2}\right\}$ (b) $\left\{\frac{1}{3}, \frac{2}{9}\right\}$ (c) $\left\{\frac{2}{9}\right\}$ (d) $\left\{\frac{1}{2}, \frac{2}{9}\right\}$

8. The minimum value of the expression $|x - p| + |x - 15| + |x - p - 15|$ for 'x' in the range $p \le x \le 15$ where $0 < p < 15$, is
(a) 10 (b) 15 (c) 30 (d) 0

9. If x, y, z are real such that $x + y + z = 4$, $x^2 + y^2 + z^2 = 6$, then the range of x is
(a) $(-1, 1)$ (b) $[0, 2]$
(c) $[2, 3]$ (d) $[2/3, 2]$

10. The roots of the equation $a(x - b)(x - c) + b(x - c)(x - a) + c(x - a)(x - b) = 0$ (a, b, c are distinct and real) are always:
(a) positive (b) negative
(c) real (d) unreal

11. If $\tan\theta$ and $\cot\theta$ are the roots of the equation $x^2 + 2x + 1 = 0$, then the least value of $x^2 + \tan\theta x + \cot\theta = 0$, is
(a) $\frac{3}{4}$ (b) $\frac{5}{4}$ (c) $\frac{-5}{4}$ (d) $\frac{-3}{4}$

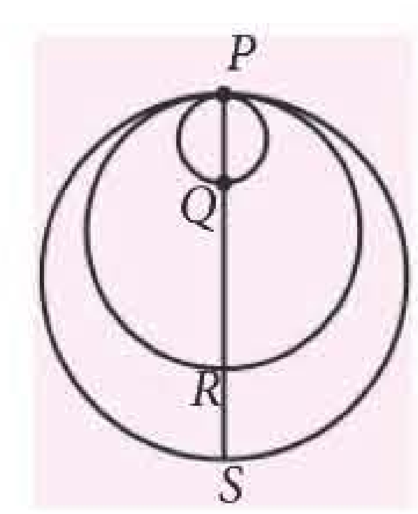
12. If one solution of the equation $x^3 - 2x^2 + ax + 10 = 0$ is the additive inverse of another, then which one of the following inequalities is true?
(a) $-40 < a < -30$ (b) $-30 < a < -20$
(c) $-20 < a < -10$ (d) $-10 < a < 0$

13. Suppose a, b and c are positive numbers such that $a + b + c = 1$, then the maximum value of $ab + bc + ca$ is
(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

14. Assume that p is a real number. In order for $\sqrt[3]{x+3p+1} - \sqrt[3]{x} = 1$ to have real solutions, it is necessary that
(a) $p \ge 1/4$ (b) $p \ge -1/4$
(c) $p \ge 1/3$ (d) $p \ge -1/3$

15 Let α, β, γ are roots of the equation $x^3 + qx + q = 0$, then find the value of $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1}$.
(a) 0 (b) -1 (c) 1 (d) none

16. $PQRS$ is a common diameter of three circles. The area of the middle circle is the average of the area of the other two. If $PQ = 2$ and $RS = 1$ then the length QR is



(a) $\sqrt{6}+1$ (b) $\sqrt{6}-1$
(c) 5 (d) 4

17. If every solution of the equation $3\cos^2 x - \cos x - 1 = 0$ is a solution of the equation $a\cos^2 2x + b\cos 2x - 1 = 0$. Then the value of $(a + b)$ is equal to
(a) 5 (b) 9 (c) 13 (d) 14

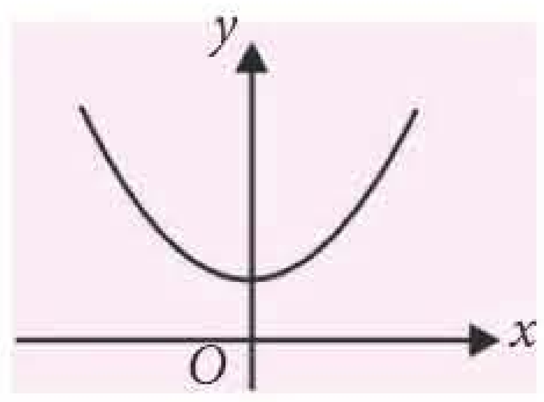
18. If all values of x obtained from the equation $4^x + (k - 3)2^x + k = 4$ are non-positive, then the largest integral value of k is
(a) 1 (b) 2 (c) 3 (d) 4

19. Let r_1, r_2 and r_3 be the solutions of the equation $x^3 - 2x^2 + 4x + 5074 = 0$ then the value of $(r_1 + 2)(r_2 + 2)(r_3 + 2)$ is
(a) 5050 (b) 5066 (c) -5050 (d) -5066

20. If α and β are the roots of the equation $(\log_2 x)^2 + 4(\log_2 x) - 1 = 0$, then the value of $\log_\beta \alpha + \log_\alpha \beta$ equals
(a) 18 (b) -16 (c) 14 (d) -18

21. Let $m(b)$ be the minimum value of $f(x) = (2 + b + b^2)x^2 - 2\sqrt{2}\,(2b + 1)x + 8$, where $b \in [-3, 10]$. The maximum value of $m(b)$ is
(a) 2 (b) 4 (c) 6 (d) 8

22. The graph of a quadratic polynomial $y = ax^2 + bx + c$ $(a, b, c \in R)$ with vertex on y-axis is as shown in the figure. Then which one of the following statement is INCORRECT?



(a) Product of the roots of the corresponding quadratic equation is positive.
(b) Discriminant of the quadratic equation is negative.
(c) Both (a) and (b)
(d) None of these

23. The quadratic equation $x^2 - 1088x + 295680 = 0$ has two positive integral roots whose greatest common divisor is 16. The least common multiple of the two roots is
(a) 18240 (b) 18480
(c) 18960 (d) 19240

24. Given a, b, c are non negative real numbers and if $a^2 + b^2 + c^2 = 1$, then the value of $a + b + c$ is
(a) ≥ 3 (b) ≥ 2 (c) $\leq \sqrt{2}$ (d) $\leq \sqrt{3}$

25. The set of values of 'a' for which the inequality, $(x - 3a)(x - a - 3) < 0$ is satisfied for all $x \in [1, 3]$ is
(a) $(1/3, 3)$ (b) $(0, 1/3)$
(c) $(-2, 0)$ (d) $(-2, 3)$

26. If the roots of the cubic, $x^3 + ax^2 + bx + c = 0$ are three consecutive positive integers. Then the value of $\dfrac{a^2}{b+1}$ is equal to
(a) 3 (b) 2 (c) 1 (d) 1/3

27. If x be the real number such that $x^3 + 4x = 8$, then the value of the expression $x^7 + 64x^2$ is
(a) 124 (b) 125 (c) 128 (d) 132

28. If the roots of the equation $x^3 - px^2 - r = 0$ are $\tan\alpha$, $\tan\beta$ and $\tan\gamma$, then the value of $\sec^2\alpha \cdot \sec^2\beta \cdot \sec^2\gamma$ is
(a) $p^2 + r^2 + 2rp + 1$ (b) $p^2 + r^2 - 2rp + 1$
(c) $p^2 - r^2 - 2rp + 1$ (d) None

29. Number of quadratic equations with real roots which remain unchanged even after squaring their roots, is
(a) 1 (b) 2 (c) 3 (d) 4

30. For a, b, c non-zero, real distinct, the equation, $(a^2 + b^2)x^2 - 2b(a + c)x + b^2 + c^2 = 0$ has non-zero real roots. One of these roots is also the root of the equation
(a) $a^2x^2 - a(b - c)x + bc = 0$
(b) $a^2x^2 + a(c - b)x - bc = 0$
(c) $(b^2 + c^2)x^2 - 2a(b + c)x + a^2 = 0$
(d) $(b^2 - c^2)x^2 + 2a(b - c)x - a^2 = 0$

31. If roots of the quadratic equation $x^2 + c = bx$ are two consecutive integers, then $b^2 - 4c$ equals
(a) -1 (b) 2 (c) 1 (d) 0

32. If the equation $\sin^4 x - (k + 2)\sin^2 x - (k + 3) = 0$ has a solution, then k must lie in the interval
(a) $(-4, -2)$ (b) $[-3, 2)$
(c) $(-4, -3)$ (d) $[-3, -2]$

33. Number of solutions of the equation $\sqrt{x^2} - \sqrt{(x-1)^2} + \sqrt{(x-2)^2} = \sqrt{5}$, is
(a) 0 (b) 1
(c) 2 (d) More than 2

34. Let a, b, c be three real numbers such that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 2$. Then the value of $(a^4 + b^4 + c^4)$ is equal to
(a) 2 (b) 5 (c) 6 (d) 8

35. A solution of the equation $4^x + 4 \cdot 6^x = 5 \cdot 9^x$ is
(a) -1 (b) 1 (c) 2 (d) 0

36. Consider two quadratic expressions $f(x) = ax^2 + bx + c$ and $g(x) = ax^2 + px + q$, $(a, b, c, p, q \in R, b \neq p)$ such that their discriminants are equal. If $f(x) = g(x)$ has a root $x = \alpha$, then
(a) α will be A.M. of the roots of $f(x) = 0$ and $g(x) = 0$
(b) α will be A.M. of the roots of $f(x) = 0$
(c) α will be A.M. of the roots of $f(x) = 0$ or $g(x) = 0$
(d) α will be A.M. of the roots of $g(x) = 0$

37. Consider the two functions $f(x) = x^2 + 2bx + 1$ and $g(x) = 2a(x + b)$, where the variable x and the constants a and b are real numbers. Each such pair of the constants a and b may be considered as a point (a, b) in an ab – plane. Let S be the set of such points (a, b) for which the graphs of $y = f(x)$ and $y = g(x)$ do not intersect (in the xy – plane.). The area of S is
(a) 1 (b) π (c) 4 (d) 4π

38. The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeroes, the product of its zeroes, and the sum of its coefficients are all equal. If the y-intercept of the graph of $y = P(x)$ is 2, then the value of b is

(a) -11 (b) -9 (c) -7 (d) 5

39. A quadratic equation, product of whose roots x_1 and x_2 is equal to 4 and satisfying the relation $\frac{x_1}{x_1-1}+\frac{x_2}{x_2-1}=2$ is

(a) $x^2 - 2x + 4 = 0$ (b) $x^2 - 4x + 4 = 0$
(c) $x^2 + 2x + 4 = 0$ (d) $x^2 + 4x + 4 = 0$

40. Number of values of x satisfying the pair of quadratic equations $x^2 - px + 20 = 0$ and $x^2 - 20x + p = 0$ for some $p \in R$, is

(a) 1 (b) 2 (c) 3 (d) 4

SOLUTIONS

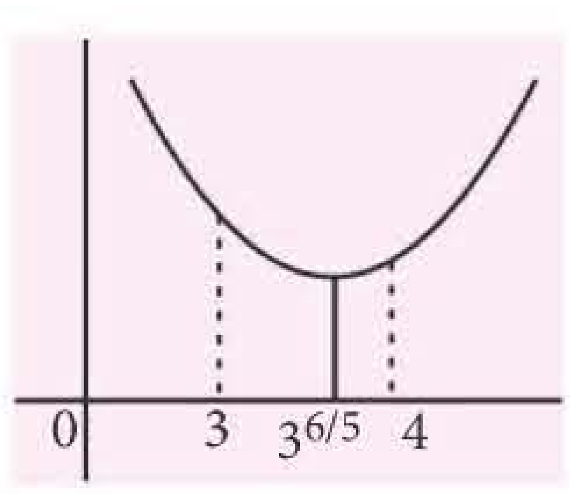
1. **(d) :** Let $\log_3 n = x$

$\therefore\ y = 5x^2 - 12x + 9$

y is minimum at $x = -\frac{b}{2a} = \frac{12}{10} = \frac{6}{5}$

Here $\log_3 n = \frac{6}{5} \Rightarrow n = 3^{6/5} \cong 3.70$

which is not natural. Hence minimum occurs at the closest integer. Now $4 > 3^{6/5}$



$\Rightarrow\ 4^5 > 3^6$

$\Rightarrow\ 1024 > 729$, which is true

2. **(c) :** If roots of equation $x^2 + px + q = 0$ are α and β, then $\alpha + \beta = -p$ and $\alpha\beta = q$

and if roots of equation $x^2 + px - r = 0$ are γ, δ then $\gamma + \delta = -p$, $\gamma\delta = -r$

Now, $(\alpha - \gamma)(\alpha - \delta) = \alpha^2 - (\gamma + \delta)\alpha + \gamma\delta = \alpha^2 + p\alpha - r$
$= -(q + r)$ $\{\because \alpha^2 + p\alpha + q = 0 \Rightarrow \alpha^2 + p\alpha = -q\}$.

3. **(b) :** $\alpha + \beta = -3;\ \alpha\beta = 1$.

$\alpha^2 + 3\alpha + 1 = 0$ and $\beta^2 + 3\beta + 1 = 0$, where

$\alpha^2 = -(3\alpha + 1)$ and $\beta^2 = -(3\beta + 1)$

Let $E = \frac{\alpha^2}{(1+\beta)^2}+\frac{\beta^2}{(\alpha+1)^2} = \frac{\alpha^2}{1+2\beta+\beta^2}+\frac{\beta^2}{1+2\alpha+\alpha^2}$

$= \left(\frac{-(3\alpha+1)}{-\beta}\right)+\left(\frac{-(1+3\beta)}{-\alpha}\right)$

$= \frac{1+3\alpha}{\beta}+\frac{1+3\beta}{\alpha} = \frac{\alpha(1+3\alpha)+\beta(1+3\beta)}{\alpha\beta}$ (as $\alpha\beta = 1$)

$= 3(\alpha^2 + \beta^2) + (\alpha + \beta) = 3(9 - 2) + (-3)$
$= 21 - 3 = 18$.

4. **(b) :** If roots are real and distinct, then $\Delta > 0$

$\therefore$ For equation $x^2 + 6x + a = 0$.

$36 - 4a > 0$ or $a < 9$ ...(i)

Also, $\alpha - \beta \le 4 \Rightarrow (\alpha - \beta)^2 \le 16$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta \le 16$

$\Rightarrow 4a \ge 20 \Rightarrow a \ge 5$.

5. **(d) :** $\cot^4 x - 2(1 + \cot^2 x) + a^2 = 0$

$\Rightarrow \cot^4 x - 2\cot^2 x + a^2 - 2 = 0$

$\Rightarrow (\cot^2 x - 1)^2 = 3 - a^2$

To have atleast one solution $3 - a^2 \ge 0$

$\Rightarrow a^2 - 3 \le 0 \Rightarrow a \in [-\sqrt{3}, \sqrt{3}]$

Integral values of a are $-1, 0, 1$

$\therefore$ Required sum = 0.

6. **(b) :** $4x^2 - 4(5x + 1) + p^2 = 0$

$\Rightarrow 4x^2 - 20x + (p^2 - 4) = 0$

Now, two roots are α, $\alpha + 2$

$\therefore\ 2\alpha+2=\frac{20}{4}=5 \Rightarrow \alpha+1=\frac{5}{2} \Rightarrow \alpha=\frac{3}{2}$

and $\alpha(\alpha+2)=\frac{p^2-4}{4} \Rightarrow \frac{3}{2}\left(\frac{3}{2}+2\right)=\frac{p^2-4}{4}$

$\Rightarrow \frac{3}{2}\cdot\frac{7}{2}=\frac{p^2-4}{4} \Rightarrow 21=p^2-4$

$\Rightarrow p^2 = 25 \Rightarrow p = \pm 5$

7. **(c) :** Let the common root be α

$\therefore\ (1 - 2k)\alpha^2 - 6k\alpha - 1 = 0$

and $k\alpha^2 - \alpha + 1 = 0$

$\Rightarrow \frac{\alpha^2}{-6k-1}=\frac{\alpha}{-k-(1-2k)}=\frac{1}{-(1-2k)+6k^2}$

$\Rightarrow \frac{\alpha^2}{-(6k+1)}=\frac{\alpha}{k-1}=\frac{1}{6k^2+2k-1}$

$\Rightarrow \alpha^2=\frac{-(6k+1)}{6k^2+2k-1},\ \alpha=\frac{k-1}{6k^2+2k-1}$

$\Rightarrow (k - 1)^2 = -(6k + 1)(6k^2 + 2k - 1)$

$\Rightarrow -k^2 + 2k - 1 = 36k^3 + 12k^2 - 6k + 6k^2 + 2k - 1$

$\Rightarrow 36k^3 + 19k^2 - 6k = 0 \Rightarrow k(36k^2 + 19k - 6) = 0$

$\because\ k \ne 0\ \therefore\ 36k^2 + 19k - 6 = 0$

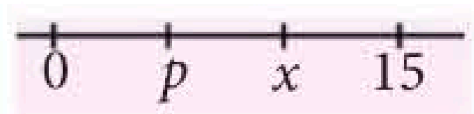
$\Rightarrow 36k^2 + 27k - 8k - 6 = 0$

$\Rightarrow 9k(4k + 3) - 2(4k + 3) = 0$

$\Rightarrow (4k + 3)(9k - 2) = 0$

$\Rightarrow k = \frac{-3}{4},\ k = \frac{2}{9}$

8. (b) : $|x - p| = x - p$ (Since $x \geq p$)
$|x - 15| = 15 - x$ (Since $x \leq 15$)
$|x - (p + 15)| = (p + 15) - x$ (Since $15 + p \geq x$)
$\therefore$ The given expression reduces to
$E = x - p + 15 - x + p + 15 - x$
$\Rightarrow$ $E = 30 - x$



$\therefore$ E_{min} occurs when $x = 15$
$\therefore$ $E_{min} = 15$.

9. (d) : $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$,
then $xy + yz + zx = 5$
$\therefore$ $y + z = 4 - x$ ...(i)
and $yz = 5 - x(4 - x)$ ...(ii)
Now, if y and z are roots of any quadratic, then
$f(t) = t^2 - (4 - x)t + 5 - x(4 - x)$
If t is real then $D \geq 0$
$\Rightarrow$ $(4 - x)^2 - 4[5 - x(4 - x)] \geq 0$
$\Rightarrow$ $16 + x^2 - 8x - 20 + 4x(4 - x) \geq 0$
$\Rightarrow$ $16 + x^2 - 8x - 20 + 16x - 4x^2 \geq 0$
$\Rightarrow$ $-3x^2 + 8x - 4 \geq 0 \Rightarrow 3x^2 - 8x + 4 \leq 0$
$$\Rightarrow \quad x \in \left[\frac{2}{3}, 2\right]$$

10. (c) : $a[x^2 - (b + c)x + bc] + b[x^2 - (c + a)x + ac] + c[x^2 - (a + b)x + ab] = 0$
$\Rightarrow$ $(a + b + c)x^2 - 2x(ab + bc + ca) + 3abc = 0$
Now, $D = 4(ab + bc + ca)^2 - 12abc(a + b + c)$
$= 4[a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c) - 3abc(a + b + c)]$
$= 4[a^2b^2 + b^2c^2 + c^2a^2 - abc(a + b + c)]$
$= 2[(ab - bc)^2 + (bc - ca)^2 + (ca - ab)^2] > 0$

11. (c) : If $\tan\theta$, $\cot\theta$ are roots of the equation $x^2 + 2x + 1 = 0$, then
$\tan\theta + \cot\theta = -2$, then $\tan\theta = -1$ and $\cot\theta = -1$
The least value of $x^2 + \tan\theta x + \cot\theta$
$$= x^2 - x - 1 = \left(x^2 - x + \frac{1}{4}\right) - \frac{5}{4}$$
$$= \left(x - \frac{1}{2}\right)^2 + \left(-\frac{5}{4}\right)$$
$\therefore$ Least value of function is $-\dfrac{5}{4}$

12. (d) : If α, β, γ are the roots of the equation, then $\alpha + \beta + \gamma = 2$. Also, $\alpha + \beta = 0$ (where α, β are additive inverse)
$\therefore$ $\gamma = 2$ which must satisfy the given equation
$\therefore$ $a = -5$

13. (a) : $a^2 + b^2 + c^2 = 1 - 2\Sigma ab$ ...(i)
Hence $a^2 + b^2 + c^2 \geq ab + bc + ca$
$\Rightarrow$ $1 - 2\Sigma ab \geq \Sigma ab$ [Using (i)]
$\Rightarrow$ $1 \geq 3\Sigma ab \therefore \Sigma ab \leq \dfrac{1}{3}$

14. (b) : We have, $\sqrt[3]{x + 3p + 1} = \sqrt[3]{x} + 1$, Let $\sqrt[3]{x} = h$
$\Rightarrow$ $\sqrt[3]{x + 3p + 1} = h + 1$
$\Rightarrow$ $x + 3p + 1 = h^3 + 3h^2 + 3h + 1$
$\Rightarrow$ $h^3 + 3p + 1 = h^3 + 3h^2 + 3h + 1$
$\Rightarrow$ $3h^2 + 3h - 3p = 0$
For real solution $D \geq 0$
$\therefore$ $b^2 - 4ac = 1 + 4p \geq 0$
or $p \geq -1/4$

Alternate solution :
$\sqrt[3]{x + 3p + 1} + (-\sqrt[3]{x}) + (-1) = 0$
If $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$
$\Rightarrow$ $x + 3p + 1 - x - 1 = 3[(x + 3p + 1)(x)]^{1/3}$
$\Rightarrow$ $3p = 3[(x + 3p + 1)(x)]^{1/3}$
$\Rightarrow$ $p^3 = x(x + 3p + 1)$
$\therefore$ $x^2 + (3p + 1)x - p^3 = 0$
For real roots $D \geq 0$
$\Rightarrow$ $4p^3 + 9p^2 + 6p + 1 \geq 0$
$\Rightarrow$ $(p + 1)^2(4p + 1) \geq 0 \Rightarrow p \geq -1/4$.

15. (c) : Given α, β, γ are roots of the equation $x^3 + qx + q = 0$
Now, $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1}$
$$= -\frac{1}{\gamma} - \frac{1}{\alpha} - \frac{1}{\beta} = -\left(\frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}\right) = 1$$
$\{\because \alpha\beta + \beta\gamma + \gamma\alpha = q, \alpha\beta\gamma = -q\}$

16. (b) : Let $QR = x$
Then the diameters are $2, x + 2, x + 3$
$$\Rightarrow \quad \frac{2^2 + (x + 3)^2}{2} = (x + 2)^2$$
$\Rightarrow$ $2(x + 2)^2 = 2^2 + (x + 3)^2$
$\Rightarrow$ $2(x^2 + 4 + 4x) = 4 + (x^2 + 6x + 9)$
$\Rightarrow$ $x^2 + 2x - 5 = 0 \Rightarrow x = \sqrt{6} - 1$

17. (c) : From 1st equation, $\cos x = \dfrac{1 \pm \sqrt{13}}{6}$
$$\text{Now, } \cos 2x = 2\cos^2 x - 1 = \frac{2}{36}(1 + 13 \pm 2\sqrt{13}) - 1$$
$$= \frac{14 \pm 2\sqrt{13} - 18}{18} = \frac{\pm 2\sqrt{13} - 4}{18} = \frac{\pm\sqrt{13} - 2}{9}$$

$$\therefore \quad \cos 2x_1 = \frac{\sqrt{13}-2}{9} \text{ and } \cos 2x_2 = -\frac{(\sqrt{13}+2)}{9}$$

Now from 2^{nd} equation,
$a\cos^2 2x + b\cos 2x - 1 = 0$

$$\therefore \quad \cos 2x_1 \cdot \cos 2x_2 = -\frac{1}{a}$$

$$\therefore \quad -\frac{1}{a} = -\left(\frac{13-4}{81}\right) = -\frac{1}{9} \quad \Rightarrow \quad a = 9$$

$$\text{and } \cos 2x_1 + \cos 2x_2 = -\frac{b}{a} = -\frac{b}{9}$$

$$\therefore \quad -\frac{b}{9} = -\frac{4}{9} \Rightarrow b = 4$$

$\therefore \quad a + b = 13.$

18. (c) : Let $2^x = t$, then
$t^2 + (k-3)t + (k-4) = 0$

$$\Rightarrow \quad t = \frac{-(k-3) \pm \sqrt{(k-3)^2 - 4(k-4)}}{2}$$

$$\Rightarrow t = \frac{-(k-3) \pm (k-5)}{2} \quad \Rightarrow t = \frac{-2k+8}{2} \quad \Rightarrow t = -k+4$$

$\therefore \quad x$ is non positive, then $x \le 0$
$\therefore \quad 0 < 2^x \le 1$
$\Rightarrow 0 < -k + 4 \le 1 \Rightarrow 3 \le k < 4$
$\therefore$ Largest integral value of k is 3.

19. (c) : $x^3 - 2x^2 + 4x + 5074 = (x - r_1)(x - r_2)(x - r_3)$
Put $x = -2$
$\therefore \quad -8 - 8 - 8 + 5074 = -(2 + r_1)(2 + r_2)(2 + r_3)$
$\therefore \quad 5050 = -(2 + r_1)(2 + r_2)(2 + r_3)$
or $\quad (2 + r_1)(2 + r_2)(2 + r_3) = -5050.$

20. (d) : $\log_2\alpha + \log_2\beta = -4$; $\log_2\alpha \cdot \log_2\beta = -1$

$$\text{Now } \log_\beta \alpha + \log_\alpha \beta = \frac{\log_2 \alpha}{\log_2 \beta} + \frac{\log_2 \beta}{\log_2 \alpha}$$

$$= \frac{(\log_2 \alpha)^2 + (\log_2 \beta)^2}{\log_2 \alpha \cdot \log_2 \beta}$$

$= -[(\log_2\alpha + \log_2\beta)^2 - 2\log_2\alpha \cdot \log_2\beta]$
$= -[16 + 2] = -18$

21. (d) : $f(x) = (2+b+b^2)x^2 - 2\sqrt{2}(2b+1)x + 8$

$$\text{Min value of } f(x) = \frac{-D}{4a} = \frac{-[8(2b+1)^2 - 32(2+b+b^2)]}{4(b^2+b+2)}$$

$$\Rightarrow m(b) = \frac{56}{4(b^2+b+2)}$$

Maximum value of $m(b)$ is obtained, when minimum value of $b^2 + b + 2$ is obtained.
Minimum value of $b^2 + b + 2$

$$= \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right) + 2 = \frac{1}{4} - \frac{1}{2} + 2 = \frac{7}{4}$$

$$\therefore \quad \text{Maximum value of } m(b) = \frac{56}{4 \times \frac{7}{4}} = 8$$

22. (d)

23. (b) : $x^2 - 1088x + 295680 = 0$
Let α and β be the the roots of given equation.
Also let $\alpha = 16k_1$ and $\beta = 16k_2$
(As H.C.F. of roots is 16)
Now, $\alpha\beta$ = (H.C.F (α, β)) (L.C.M (α, β))
$\Rightarrow \quad 295680 = 16(\text{L.C.M.}\ (\alpha, \beta))$

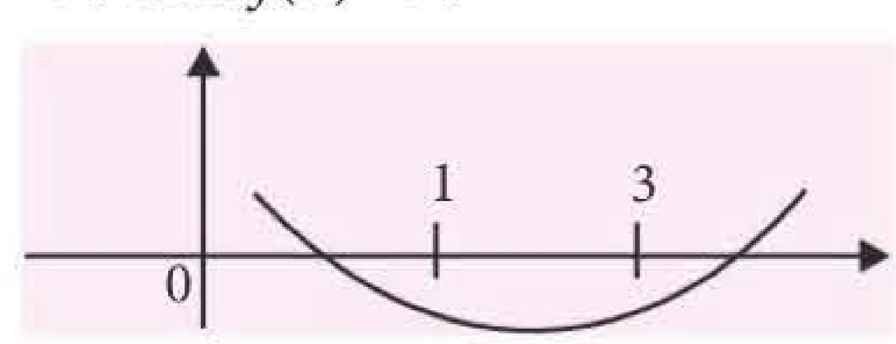
$$\Rightarrow \quad \text{L.C.M}\ (\alpha, \beta) = \frac{295680}{16} = 18480$$

24. (d) : Using R.M.S. $\ge$ AM in a, b, c, we get

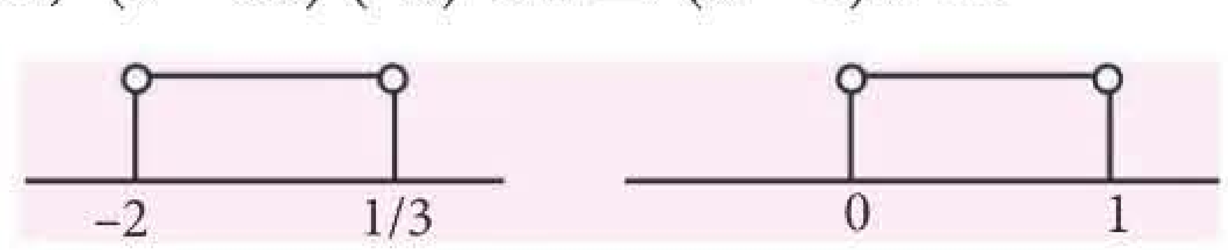
$$\sqrt{\frac{a^2+b^2+c^2}{3}} \ge \frac{a+b+c}{3}$$

$$\Rightarrow \quad \sqrt{3} \ge a+b+c \Rightarrow a+b+c \le \sqrt{3}$$

25. (b) : The given equation is $x^2 - (4a + 3)x + 3a(a + 3)$
Now, $f(1) < 0$ and $f(3) < 0$



$\Rightarrow \quad (1 - 3a)\ (1 - a - 3) < 0$
$\Rightarrow \quad 1 - a - 3 - 3a + 3a^2 + 9a < 0$
$\Rightarrow \quad 3a^2 + 5a - 2 < 0 \Rightarrow \quad 3a^2 + 6a - a - 2 < 0$
$\Rightarrow \quad 3a(a + 2) - (a + 2) < 0$
$\Rightarrow \quad (a + 2\ )\ (3a - 1) < 0$
Again, $(3 - 3a)\ (-a) < 0 \Rightarrow (a - 1)a < 0$



$\Rightarrow \quad 0 < a < 1$

$$\text{Hence, } a \in \left(0, \frac{1}{3}\right)$$

26. (a) : Let n, $n + 1$, $n + 2$ are the roots of the given equation.
$\therefore \quad$ Sum $= 3(n + 1) = -a$

$\Rightarrow a^2 = 9(n+1)^2$ ...(i)

Let sum of the roots taken 2 at a time = b

$\therefore$ $n(n+1) + (n+1)(n+2) + (n+2)(n) + 1 \;\; b + 1$ (adding 1 on both sides)

$\Rightarrow n^2 + n + n^2 + 3n + 2 + n^2 + 2n + 1 = b + 1$

$\Rightarrow b + 1 = 3n^2 + 6n + 3 = 3(n+1)^2$

$\Rightarrow b + 1 = 3(n+1)^2 = \dfrac{a^2}{3}$ [Using (i)]

$\therefore \dfrac{a^2}{b+1} = 3$

27. (c) : Given, $x^3 + 4x - 8 = 0$

Let $y = x^7 + 64x^2$

$= \underbrace{x^4(x^3+4x-8)}_{\text{zero}} - 4x^5 + 8x^4 + 64x^2$

$= -4x^5 + 8x^4 + 64x^2$

$= -4x^2\underbrace{(x^3+4x-8)}_{\text{zero}} + 8x^4 + 16x^3 + 32x^2$

$= 8x^4 + 16x^3 + 32x^2$

$= 8x\underbrace{(x^3+4x-8)}_{\text{zero}} + 16x^3 + 64x$

$= \underbrace{16(x^3+4x-8)}_{\text{zero}} + 128 = 128$

28. (b) : $\sum \tan\alpha = p,\ \sum \tan\alpha\cdot\tan\beta = 0,\ \prod \tan\alpha = r$

Now, $\sec^2\alpha\cdot\sec^2\beta\cdot\sec^2\gamma$

$= (1+\tan^2\alpha)(1+\tan^2\beta)(1+\tan^2\gamma)$

$= 1 + \sum(\tan^2\alpha) + \sum(\tan^2\alpha\cdot\tan^2\beta) + \tan^2\alpha\cdot\tan^2\beta\cdot\tan^2\gamma$

Now, $\sum \tan^2\alpha = \left(\sum\tan\alpha\right)^2 - 2\sum\tan\alpha\cdot\tan\beta = p^2$

$\sum\tan^2\alpha\cdot\tan^2\beta = \left(\sum\tan\alpha\cdot\tan\beta\right)^2$

$-2\tan\alpha\cdot\tan\beta\cdot\tan\gamma\left(\sum\tan\alpha\right)$

$= 0 - 2rp$

Also, $\prod\tan^2\alpha = r^2$

$\therefore \prod\sec^2\alpha = 1 + p^2 - 2rp + r^2 = 1 + (p-r)^2$

29. (c) : $\alpha\beta = \alpha^2\beta^2$ ...(1)

and $\alpha^2 + \beta^2 = \alpha + \beta$ ...(2)

Hence $\alpha\beta(1-\alpha\beta) = 0 \Rightarrow \alpha = 0$ or $\beta = 0$ or $\alpha\beta = 1$

If $\alpha = 0$, then from (2), $\beta = 0$ or $\beta = 1$

$\Rightarrow$ Roots are (0, 0) or (0, 1)

If $\beta = 0$, then $\alpha = 0$ or $\alpha = 1$

$\Rightarrow$ Roots are (0, 0) or (1, 0)

If $\beta = \dfrac{1}{\alpha}$, then $\alpha^2 + \dfrac{1}{\alpha^2} = \alpha + \dfrac{1}{\alpha}$

$\Rightarrow \left[\alpha + \dfrac{1}{\alpha}\right]^2 - 2 = \alpha + \dfrac{1}{\alpha}$

Hence, $t^2 - t - 2 = 0$

$\Rightarrow (t-2)(t+1) = 0 \Rightarrow t = 2$ or $t = -1$

If $t = 2 \Rightarrow \alpha = 1$ and $\beta = 1$, if $t = -1$, then roots are imaginary (ω or ω^2).

30. (b) : Let α, β are the roots of the given equation.

$$\therefore \alpha,\beta = \frac{2b(a+c) \pm \sqrt{4b^2(a+c)^2 - 4(a^2+b^2)(b^2+c^2)}}{2(a^2+b^2)}$$

$$= \frac{b(a+c) \pm \sqrt{b^2(a^2+2ac+c^2) - (a^2b^2 + a^2c^2 + b^4 + b^2c^2)}}{a^2+b^2}$$

$$= \frac{b(a+c) \pm \sqrt{-(b^4 - 2b^2ac + a^2c^2)}}{a^2+b^2}$$

$$= \frac{b(a+c) \pm \sqrt{-(b^2-ac)^2}}{a^2+b^2}$$

In order that roots may be real $D \ge 0 \Rightarrow D = 0$

$\Rightarrow b^2 - ac = 0 \Rightarrow b^2 = ac$

Hence roots are co-incident and equal to

$\dfrac{b(a+c)}{a^2+ac} = \dfrac{b}{a}$ which satisfies B.

31. (c) : We have, $x^2 - bx + c = 0$

Let the roots are α and $\alpha + 1$

$\therefore$ Sum of roots $= 2\alpha + 1 = b$ ...(i)

Product $= \alpha(\alpha+1) = c$ ...(ii)

From (i), $\alpha = (b-1)/2$

Put the value of α in (ii), we get

$\Rightarrow \left(\dfrac{b-1}{2}\right)^2 + \left(\dfrac{b-1}{2}\right) = c$

$\Rightarrow b^2 - 2b + 1 + 2b - 2 = 4c \Rightarrow b^2 - 4c = 1$

32. (d) : $\sin^2 x = \dfrac{(k+2) \pm \sqrt{(k+2)^2 + 4(k+3)}}{2}$

$= \dfrac{(k+2) \pm \sqrt{k^2+8k+16}}{2} = \dfrac{(k+2) \pm (k+4)}{2}$

$\Rightarrow \sin^2 x = k + 3$ or -1 (rejected)

$\Rightarrow\ 0 \le k+3 \le 1 \Rightarrow -3 \le k \le -2$

33. (c) : $|x| - |x-1| + |x-2| = \sqrt{5}$

For $x \ge 2$, $x - (x-1) + x - 2 = \sqrt{5}$

$\Rightarrow\ x = 1+\sqrt{5}$

For $1 \le x < 2$, $\ x - (x-1) + (2-x) = \sqrt{5}$

$\Rightarrow\ 3 - x = \sqrt{5}$

$\Rightarrow\ x = 3-\sqrt{5}$ (No solution)

For $0 \le x < 1$, $\ x - (1-x) + 2 - x = \sqrt{5}$

$\Rightarrow\ x = \sqrt{5}-1$ (No solution)

For $x < 0$, $\ -x - (1-x) + 2 - x = \sqrt{5}$

$\Rightarrow\ x = 1-\sqrt{5}$

Hence, $x = \sqrt{5}+1$ or $x = 1-\sqrt{5}$

34. (a) : Given, $a+b+c=0$

and $a^2+b^2+c^2=2$

$\Rightarrow (a^2+b^2+c^2)^2 = 4$

$\Rightarrow a^4+b^4+c^4+2[a^2b^2+b^2c^2+c^2a^2] = 4$

Let $a^4+b^4+c^4 = E$

Hence, $E + 2[(ab+bc+ca)^2 - 2abc(a+b+c)] = 4$

$\therefore\ E + 2(ab+bc+ca)^2 = 4$ ...(i) (As $a+b+c=0$)

Again, $(a+b+c)^2 = 0 \Rightarrow \sum a^2 + 2\sum ab = 0$

$\Rightarrow 2 + 2(ab+bc+ca) = 0$

$\Rightarrow ab+bc+ca = -1$

$\therefore$ From (i), $E + 2 = 4$

$\therefore\ E = 2$

35. (d) : $4^x + 4\cdot 6^x = 5\cdot 9^x$

$$\Rightarrow \left(\frac{4}{6}\right)^x + 4 = 5\left(\frac{9}{6}\right)^x \Rightarrow \left(\frac{2}{3}\right)^x + 4 = 5\cdot\left(\frac{3}{2}\right)^x$$

Let $\left(\frac{2}{3}\right)^x = t$, then $t+4 = 5\left(\frac{1}{t}\right) \Rightarrow t^2+4t-5=0$

$\Rightarrow (t+5)(t-1) = 0$

$\Rightarrow t \ne -5$ (As t can't be negative)

$\therefore\ t = 1 \Rightarrow \left(\frac{2}{3}\right)^x = 1 \Rightarrow x = 0$ is the solution.

36. (a) : $a\alpha^2 + b\alpha + c = a\alpha^2 + p\alpha + q$ [$\because f(\alpha) = g(\alpha)$]

$\therefore\ \alpha = \dfrac{q-c}{b-p}$ ...(i)

A.M. of roots of $f(x) = 0$ and $g(x) = 0$ is

$\dfrac{-b/a - p/a}{4} = -\dfrac{b+p}{4a}$ (Number of roots is 4)

Since it is given that discriminants are equal

$\therefore\ b^2 - 4ac = p^2 - 4aq$

$\Rightarrow b^2 - p^2 = 4ac - 4aq$

$\Rightarrow \dfrac{b+p}{4a} = \dfrac{c-q}{b-p} = -\alpha$ [From (i)]

$\therefore\ \alpha =$ A.M. of roots of $f(x) = 0$ and $g(x) = 0$

37. (b) : We need $x^2+2bx+1 = 2ax+2ab$ not to have any real solutions, implying that the discriminant is less than or equal to zero. Actually calculating the discriminant and simplifying, we get $a^2+b^2<1$, which describes a circle of area π in the ab plane]

38. (a) : The y-intercept is at $x = 0$, so we have $c = 2$, meaning that the product of the roots is -2. We know that a is the sum of the roots. The average of the roots is equal to the product, so the sum of the roots is -6, and $a = 6$. Finally, $1 + a + b + c = -2$ as well, so we have $1 + 6 + b + 2 = -2 \Rightarrow b = -11$

39. (a) : Given, $x_1 x_2 = 4 \Rightarrow x_2 = \dfrac{4}{x_1}$

Consider $\dfrac{x_1}{x_1-1} + \dfrac{x_2}{x_2-1} = 2$

$$\Rightarrow \frac{x_1}{x_1-1} + \frac{\frac{4}{x_1}}{\frac{4}{x_1}-1} = 2 \Rightarrow \frac{x_1}{x_1-1} + \frac{4}{4-x_1} = 2$$

$\Rightarrow 4x_1 - x_1^2 + 4x_1 - 4 = 2(x_1-1)(4-x_1)$

$\Rightarrow x_1^2 - 2x_1 + 4 = 0 \Rightarrow x^2 - 2x + 4 = 0.$

40. (c) : $x^2 - px + 20 = 0$

and $x^2 - 20x + p = 0$

If $p = 20$, then both the quadratic equations are identical.

Hence, $x = 10+4\sqrt{5}$

or $x = 10-4\sqrt{5}$ satisfy both.

If $p \ne 20$, then $x^2 - px + 20 = x^2 - 20x + p$

$\Rightarrow (20-p)x + (20-p) = 0$

$\Rightarrow x = -1$ and $p = -21$

Hence, there are 3 values of x

i.e., $\left\{10+4\sqrt{5},\ 10-4\sqrt{5},\ -1\right\}$

GEAR UP FOR JEE MAIN

KEY POINTS

Equation of circle

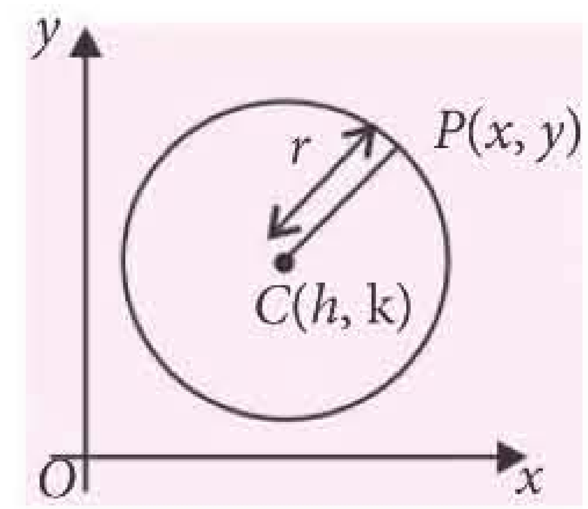
Let $C(h, k)$ be the centre of the circle and $CP(= r)$ be the radius of circle, then equation of circle is

$(x - h)^2 + (y - k)^2 = r^2$ ...(i)

Now, if origin (0, 0) be the centre of circle, then eq. (i) becomes,

$x^2 + y^2 = r^2$ ...(ii)

The area of the circle is given by πr^2 sq.unit.



General Equation of Circle

The general equation of second degree may represents a circle, if the coefficient of x^2 and coefficient of y^2 are identical and the coefficient of xy becomes zero. *i.e.*,

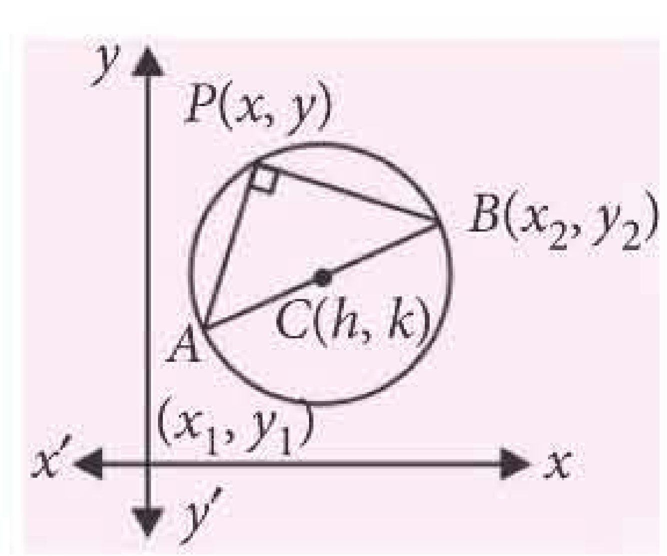
$ax^2 + by^2 + 2hxy + + 2gx + 2fy + c = 0$ ...(i)

represents a circle, if (a) $a = b$ *i.e.*, coefficient of x^2 = coefficient of y^2 and (b) $h = 0$ *i.e.*, coefficient of $xy = 0$, then Eq.(i) reduces as, $x^2 + y^2 + 2gx + 2fy + c = 0$ whose centre and radius are $(-g, -f)$ and $\sqrt{g^2 + f^2 - c}$ respectively.

- **Equation of circle in diameter form**

  Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of the given circle and let $P(x, y)$ be any point on the circle.

  $\therefore$ From figure, $\angle APB = 90°$



  $\therefore$ Slope of AP,

  $m_1 = \left(\dfrac{y - y_1}{x - x_1}\right)$ and slope of BP, $m_2 = \left(\dfrac{y - y_2}{x - x_2}\right)$

  For perpendicular, $m_1 \cdot m_2 = -1$

  $\therefore$ $AP \cdot BP = -1$

  $\Rightarrow \left(\dfrac{y - y_1}{x - x_1}\right)\left(\dfrac{y - y_2}{x - x_2}\right) = -1$

  $\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$,

  which is the required equation of circle in diameter form.

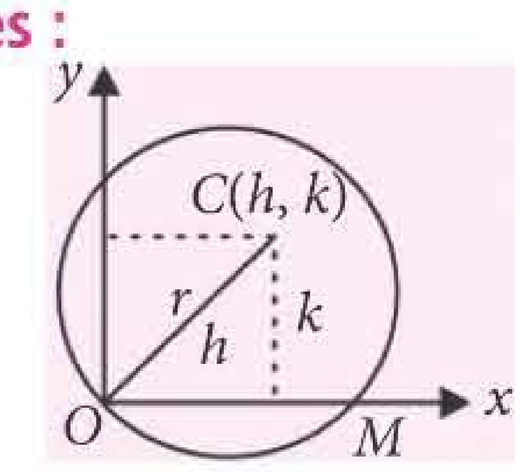
Equation of circle in different cases :

- **Case I :** When the circle passes through the origin (0, 0) : Let the equation of circle be

  $(x - h)^2 + (y - k)^2 = r^2$ ....(i)

  $\because$ It passes through origin (0, 0)

  $\therefore$ $h^2 + k^2 = r^2$



  $\therefore$ Equation (i) becomes,

  $(x - h)^2 + (y - k)^2 = h^2 + k^2$

  $\Rightarrow$ $x^2 + h^2 - 2hx + y^2 - 2ky + k^2 = h^2 + k^2$

  $\Rightarrow$ $x^2 + y^2 - 2hx - 2ky = 0$

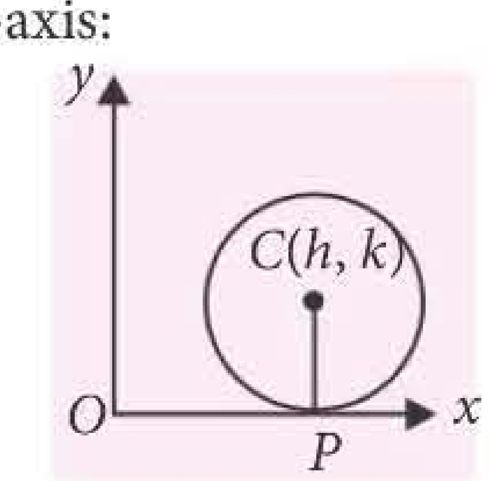
- **Case II :** When the circle touches x-axis:

  Let the centre of circle be $C(h, k)$, and it touches x- axis at point P, then the radius of circle is $CP = |k|$

  $\therefore$ Equation of circle is

  $(x - h)^2 + (y - k)^2 = (CP)^2 = k^2$

  or $x^2 + y^2 - 2hx - 2ky + h^2 = 0$



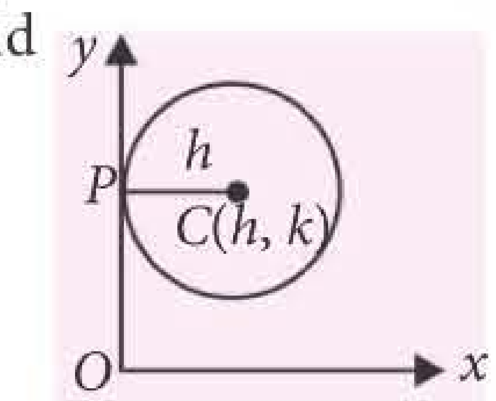
- **Case III :** When the circle touches y-axis:

  Let the centre of circle be $C(h, k)$ and it touches y-axis at point P, then the radius $CP = |h|$

  $\therefore$ Equation of circle is

  $(x - h)^2 + (y - k)^2 = (CP)^2 = h^2$

  or $x^2 + y^2 - 2hx - 2ky + k^2 = 0$



- **Case IV :** When the circle touches both axis:

  In this case $|h| = |k| = \alpha$.

  Then the equation of circle is

  $(x - h)^2 + (y - k)^2 = r^2$ where, $|h| = |k| = |r| = \alpha$

  $\therefore$ $(x \pm \alpha)^2 + (y \pm \alpha)^2 = \alpha^2$

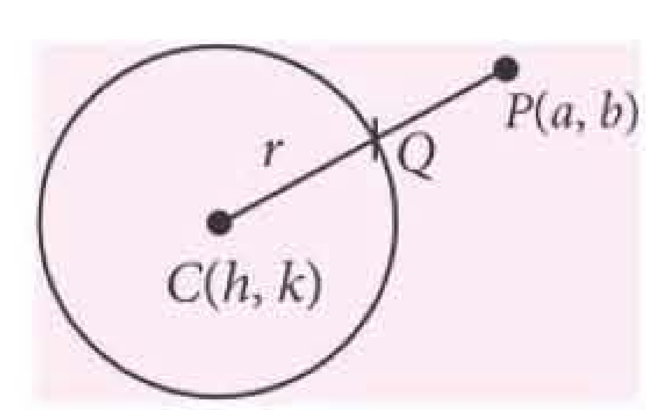
  or $x^2 + y^2 \pm 2\alpha x \pm 2\alpha y + \alpha^2 = 0$

Position of a point with respect to a circle

Let $C(h, k)$ be the centre and r be the radius of the circle and $P(a, b)$ be any point in the plane of the circle, then three cases arises *i.e.*,

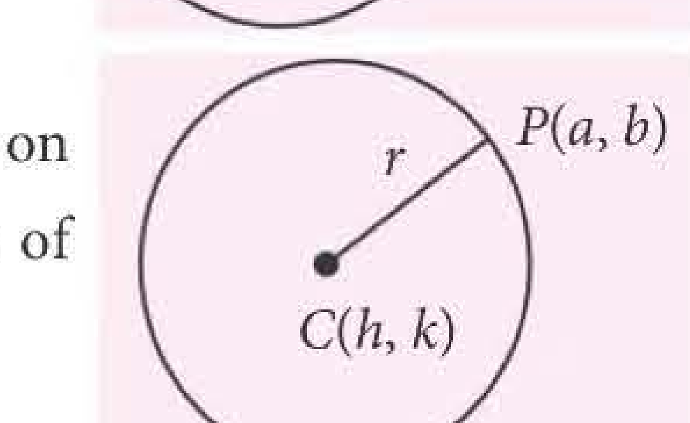
- **Case I :** Let 'P' lies outside the circle, then equation of circle is

  $(a - h)^2 + (b - k)^2 > r^2$



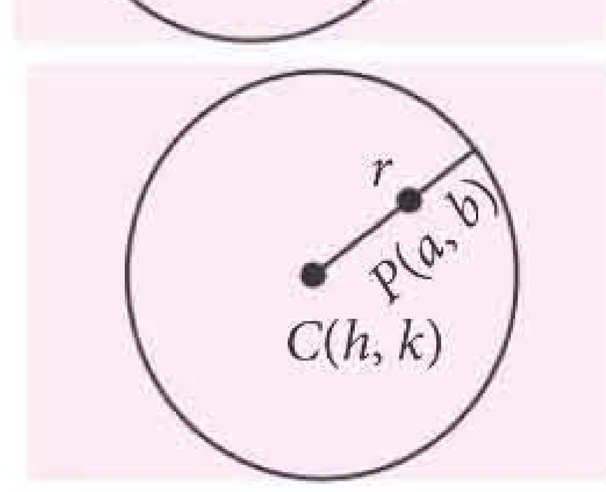
- **Case II :** Let point 'P' lies on the circle, then equation of circle is

  $(a - h)^2 + (b - k)^2 = r^2$



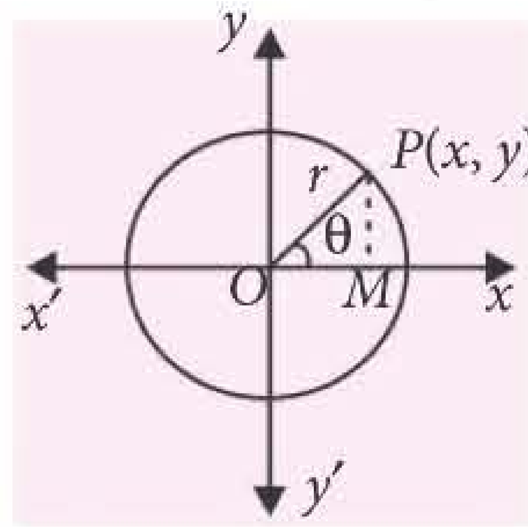
- **Case III :** Let point 'P' lies inside the circle, then equation of circle is

  $(a - h)^2 + (b - k)^2 < r^2$



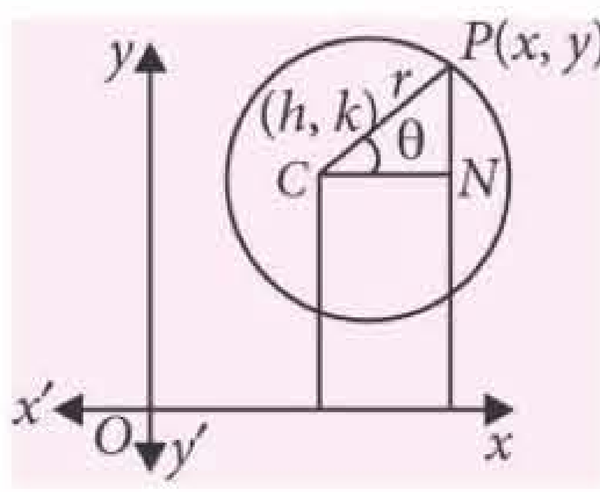
Equation of circle in parametric form

- **Case I :** Let $P(x, y)$ be any point on the circle $x^2 + y^2 = r^2$, then from fig. $\angle MOP = \theta$. On resolving the components, we get



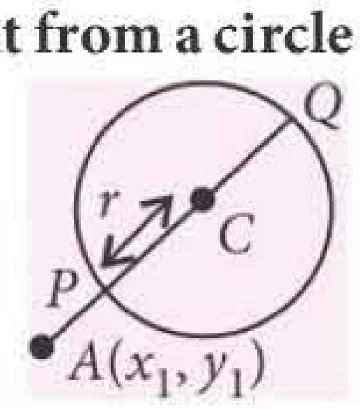
$x = OM = r\cos\theta$ ....(i)
and $y = PM = r\sin\theta$ ....(ii)
Here eqs. (i) and (ii) are the required parametric form of the circle $x^2 + y^2 = r^2$, where 'θ' is a parameter.

Case II : Parametric form of equation of circle, if (h, k) is the centre and r being the radius is



$x = h + r\cos\theta$,
$y = k + r\sin\theta$, $0 \le \theta \le 2\pi$
where θ being the parameter.

- **The least and greatest distance of a point from a circle**
Let $S = 0$ be a circle and $A(x_1, y_1)$ be a point. If the diameter of the circle is passing through the circle at P and Q, then $AP = AC - r$ = least distance.
$AQ = AC + r$ = greatest distance where r is the radius and C is the centre of circle.

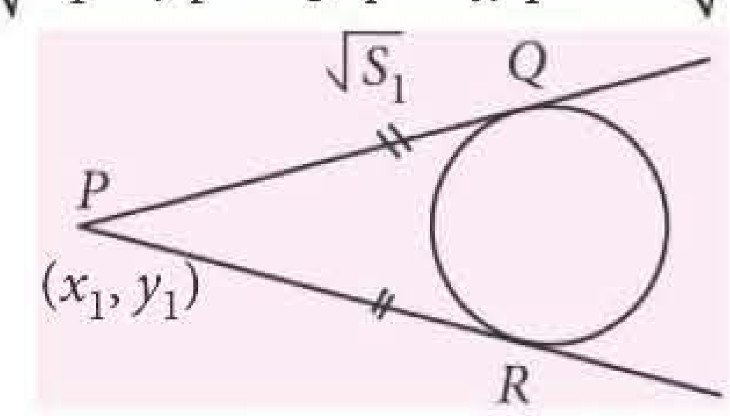


Condition of tangency

- A line $L = 0$ touches the circle $S = 0$, if length of perpendicular drawn from the centre of the circle to the line is equal to radius of the circle *i.e.*, $p = r$. This is the condition of tangency for the line $L = 0$.
Circle $x^2 + y^2 = a^2$ will touch the line $y = mx + c$ if
$c = \pm a\sqrt{1+m^2}$
(a) If $a^2(1 + m^2) - c^2 > 0$ line will meet the circle at real and different points.
(b) If $c^2 = a^2(1 + m^2)$ line will touch the circle.
(c) If $a^2(1 + m^2) - c^2 < 0$ line will meet circle at two imaginary points (*i.e.* will never meet the circle).

Equation of tangent and normal

- **Equation of tangent :** The equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at a point (x_1, y_1) is
$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$
or $T = 0$
The equation of tangent to circle $x^2 + y^2 = a^2$ at point (x_1, y_1) is $xx_1 + yy_1 = a^2$.
- **Slope form :** From condition of tangency for every value of m, the line $y = mx \pm a\sqrt{1+m^2}$ is a tangent of the circle $x^2 + y^2 = a^2$ and its point of contact is
$$\left(\frac{\mp am}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$$
- **Equation of normal :** Normal to a curve at any point P of a curve is the straight line passing through P and is perpendicular to the tangent at P. The equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) is $y - y_1 = \dfrac{y_1 + f}{x_1 + g}(x - x_1)$
- **Length of tangent :** From any point, say $P(x_1, y_1)$ two tangents can be drawn to a circle which are real, coincident or imaginary according as P lies outside, on or inside the circle.
Let PQ and PR be the two tangents drawn from $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then $PQ = PR$ is called the length of tangent drawn from point P and is given by
$$PQ = PR = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$$



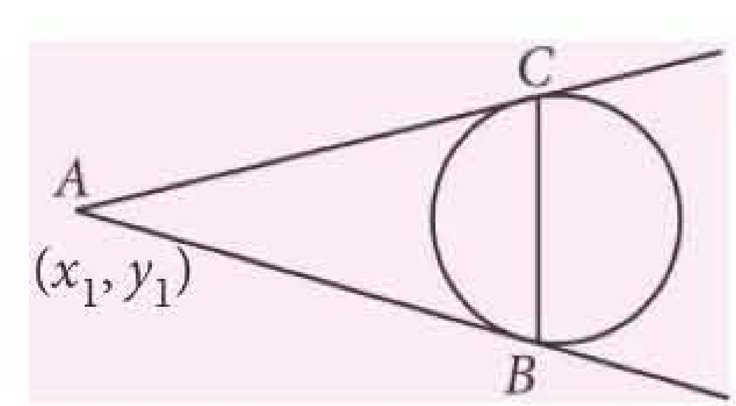
- **Pair of tangents :** From a given external point $P(x_1, y_1)$ two tangents PQ and PR can be drawn to the circle, $S = x^2 + y^2 + 2gx + 2fy + c = 0$.
Their combined equation is $SS_1 = T^2$, where $S = 0$ is the equation of circle, $T = 0$ is the equation of the tangent at (x_1, y_1) and S_1 is obtained by replacing x by x_1 and y by y_1 in S.

Director circle

The locus of the point of intersection of two perpendicular tangents to a circle is called the Director circle. Let the circle be $x^2 + y^2 = a^2$, then equation of pair of tangents to a circle from a point (x_1, y_1) is
$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$
If this represents a pair of perpendicular lines then coefficient of x^2 + coefficient of $y^2 = 0$
i.e., $(x_1^2 + y_1^2 - a^2 - x_1^2) + (x_1^2 + y_1^2 - a^2 - y_1^2) = 0$
$\Rightarrow \ x_1^2 + y_1^2 = 2a^2$
Hence the equation of director circle is $x^2 + y^2 = 2a^2$.
Obviously, director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the given circle.
Director circle of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is
$$x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$$

Chord of contact

The chord joining the two points of contact of tangents to a circle drawn from any external point A is called chord of contact of A with respect to the given circle. Let the given point is $A(x_1, y_1)$ and the circle is $S = 0$ then equation of the chord of contact is



$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ ... (i)

Note : (i) It is clear from the above that the equation of the chord of contact coincides with the equation of the tangent, if the point (x_1, y_1) lies on the circle.

(ii) The length of chord of contact = $2\sqrt{r^2 - p^2}$

(iii) Area of $\Delta ABC = \dfrac{a(x_1^2 + y_1^2 - a^2)^{3/2}}{x_1^2 + y_1^2}$

Equation of a chord whose middle point is given

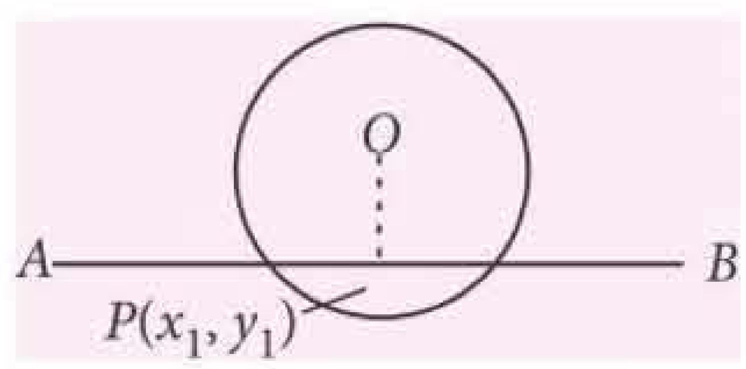
We have the circle $x^2 + y^2 = a^2$ and middle point of chord is $P(x_1, y_1)$.

Slope of the line $OP = \dfrac{y_1}{x_1}$; slope of $AB = -\dfrac{x_1}{y_1}$

So equation of chord is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

or $xx_1 + yy_1 = x_1^2 + y_1^2$



which can be represented by $T = S_1$.

Common chord of two circles

The line joining the points of intersection of two circles is called the common chord. If the equation of two circles is

$$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$
$$S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

then equation of common chord is

$S_1 - S_2 = 0 \Rightarrow 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

The length of the common chord is

$$2\sqrt{r_1^2 - p_1^2} = 2\sqrt{r_2^2 - p_2^2}$$

where p_1 and p_2 are the length of perpendicular drawn from the centre to the chord.

Angle of intersection of two circles

The angle of intersection between two circles $S = 0$ and $S' = 0$ is defined as the angle between their tangents at their point of intersection. If

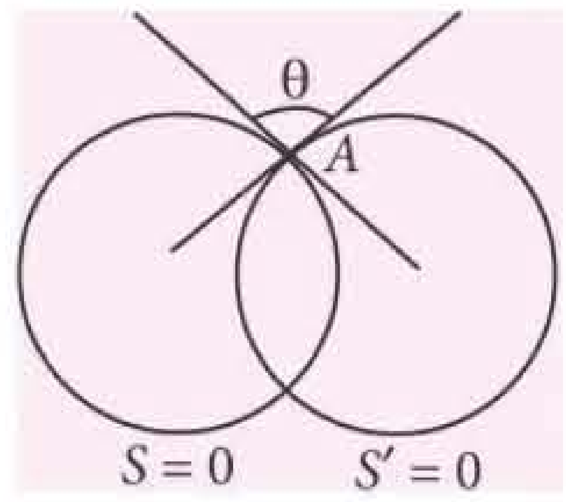
$$S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$
$$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

are two circles with radii r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

or $$\cos\theta = \frac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}}$$

- **Condition of orthogonality :** If the angle of intersection of the two circles is 90° then such circles are called orthogonal circles and condition for orthogonal circles and condition for orthogonality is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$. When the two circles intersect orthogonally then the length of tangent on one circle from the centre of other circle is equal to the radius of the other circle.



Power of a point with respect to a circle

The power of a point $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is S_1 where

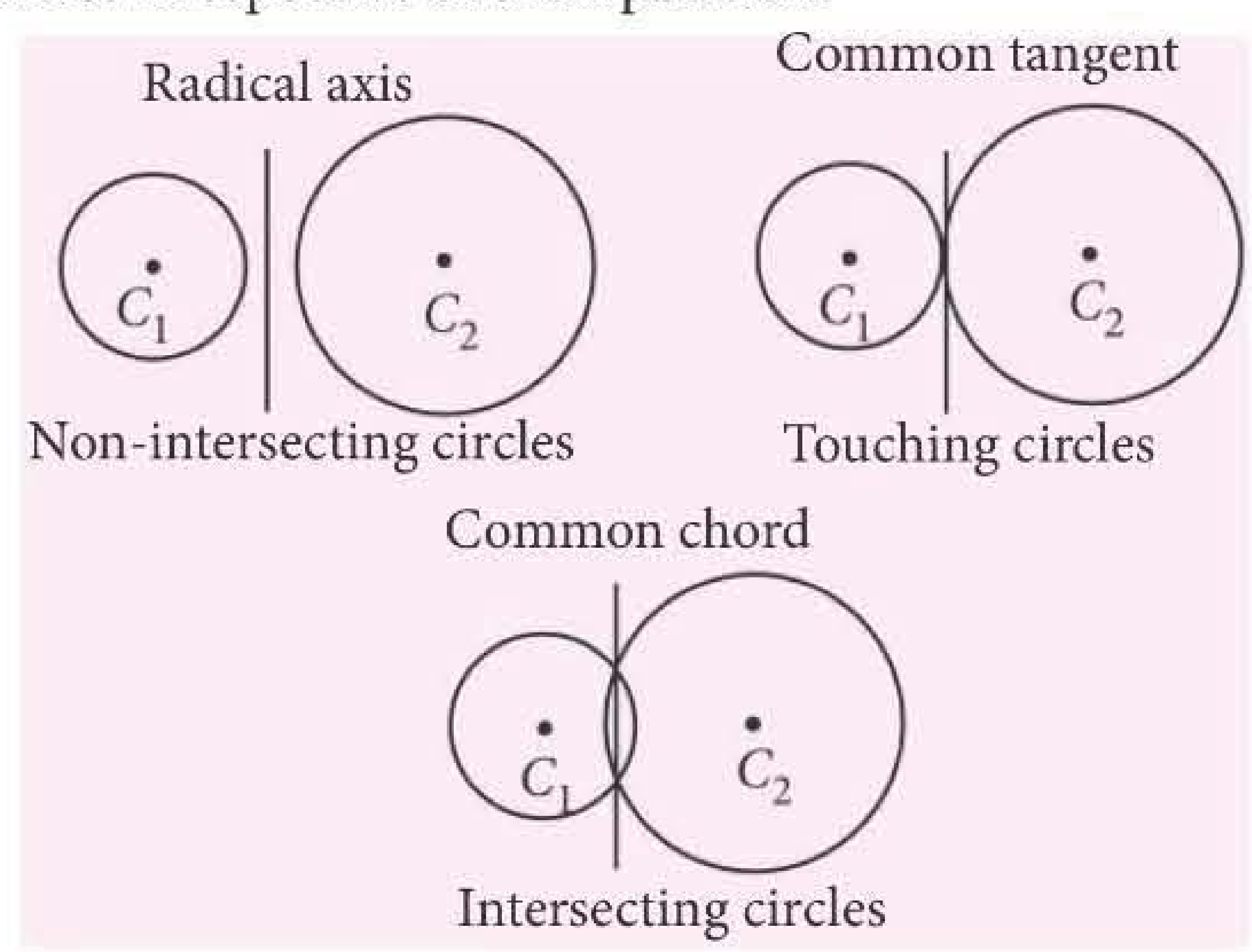
$$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 = 0$$

Radical axis

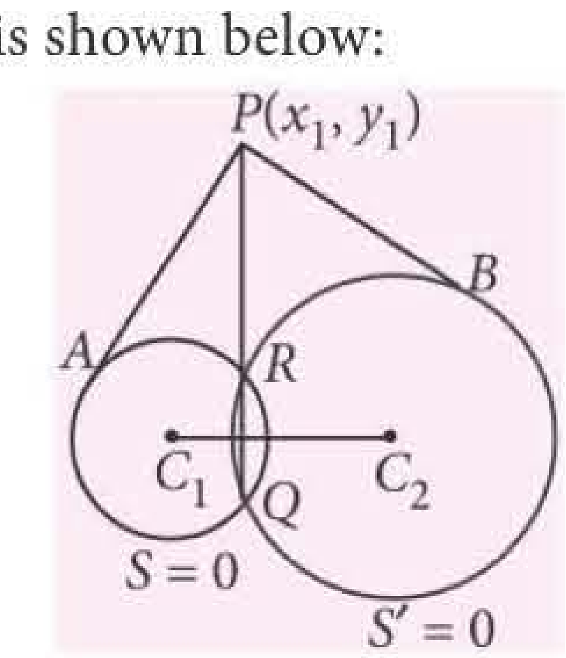
The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal in length.

Some properties of the radical axis are as follows :

- **The radical axis and common chord are identical** : Since the radical axis and common chord of the two circles $S = 0$ and $S' = 0$ are the same straight line $S - S' = 0$, they are identical. The only difference is that the common chord exists only if the circles intersect in two real points, while the radical axis exists for all pair of circles irrespective of their position.



The position of the radical axis of the two circles geometrically is shown below:



From Euclidean geometry, $(PA)^2 = PR \cdot PQ = (PB)^2$

- The radical axis is perpendicular to the straight line which joins the centres of the circles.

Consider, $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ ... (i)

and $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ... (ii)

Since $C_1 \equiv (-g, -f)$ and $C_2 \equiv (-g_1, -f_1)$ are the centres of the circles (i) and (ii), then slope of

$$C_1C_2 = \frac{-f_1 + f}{-g_1 + g} = \frac{f - f_1}{g - g_1} = m_1 \text{ (say)}$$

Equation of the radical axis is

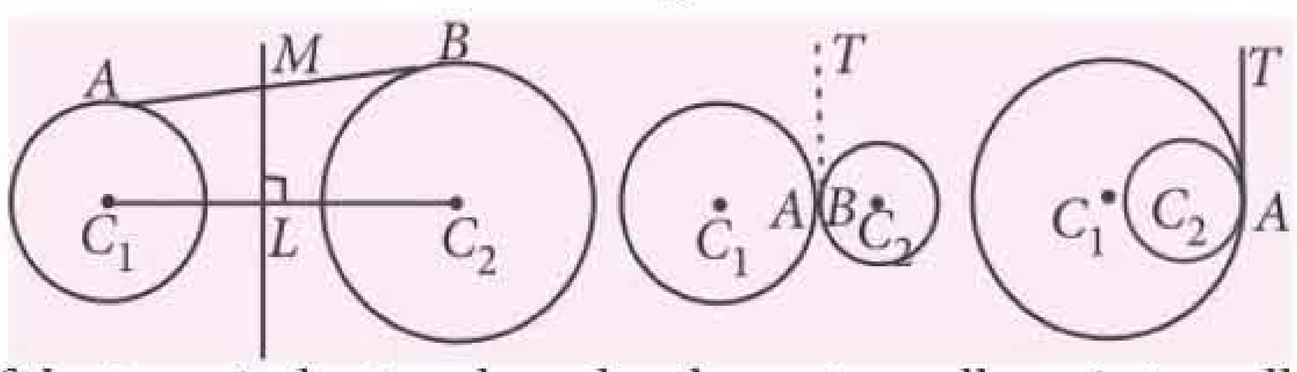
$2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$

Slope of radical axis is $\dfrac{-(g - g_1)}{f - f_1} = m_2$ (say)

$\because\ m_1m_2 = -1$

Hence C_1C_2 and radical axis are perpendicular to each other.

- **The radical axis bisects common tangents of two circles:** Let AB be the common tangent. If it meets the radical axis LM at M, then MA and MB are two tangents to the circles. Hence $MA = MB$ since lengths of tangents are equal from any point on radical axis. Hence radical axis bisects the common tangent AB.



If the two circles touch each other externally or internally, then A and B coincides. In this case the common tangent itself becomes the radical axis.

- **The radical axis of three circles taken in pairs are concurrent :** Let the equation of three circles be

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ... (i)

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ... (ii)

$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$ ... (iii)

The radical axis of the above three circles taken in pairs are given by

$S_1 - S_2 \equiv 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ ...(iv)

$S_2 - S_3 \equiv 2x(g_2 - g_3) + 2y(f_2 - f_3) + c_2 - c_3 = 0$ ... (v)

$S_3 - S_1 \equiv 2x(g_3 - g_1) + 2y(f_3 - f_1) + c_3 - c_1 = 0$ ... (vi)

Adding (iv), (v) and (vi), we find LHS vanished identically. Thus the three lines are concurrent.

- If two circles cut the third circle orthogonally, then the radical axis of the two circles will pass through the centre of the third circle.

OR

The locus of the centre of a circle cutting two given circles orthogonally is the radical axis of the two circles.

Let $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ...(i)

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ...(ii)

$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$ ...(iii)

Since (i) and (ii) both cut (iii) orthogonally

$\therefore \quad 2g_1g_3 + 2f_1f_3 = c_1 + c_3$

and $2g_2g_3 + 2f_2f_3 = c_2 + c_3$

Subtracting, we get

$2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2$ ...(iv)

Now radical axis of (i) and (ii) is

$S_1 - S_2 = 0$ or $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

Since it will pass through the centre of (iii) circle

$\therefore \quad -2g_3(g_1 - g_2) - 2f_3(f_1 - f_2) + c_1 - c_2 = 0$

or $\quad 2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2$ ...(v)

which is true by (iv).

Some important results to remember

- If two conic sections

$a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

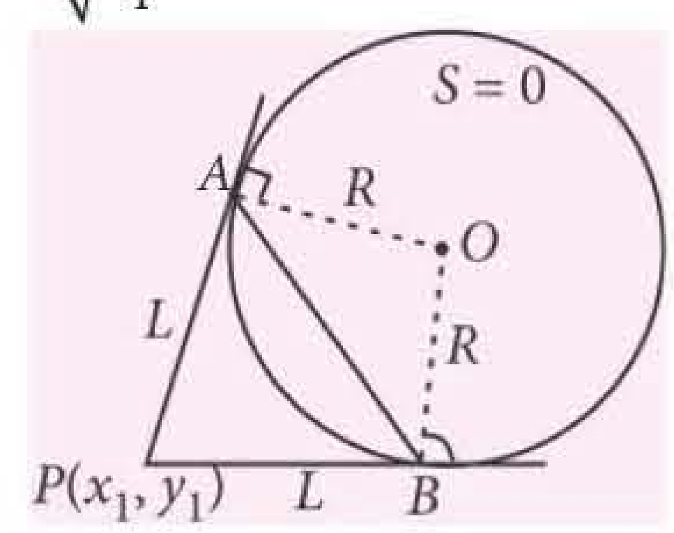
$a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0$ will

intersect each other in four concyclic points, if $\dfrac{a_1 - b_1}{a_2 - b_2} = \dfrac{h_1}{h_2}$.

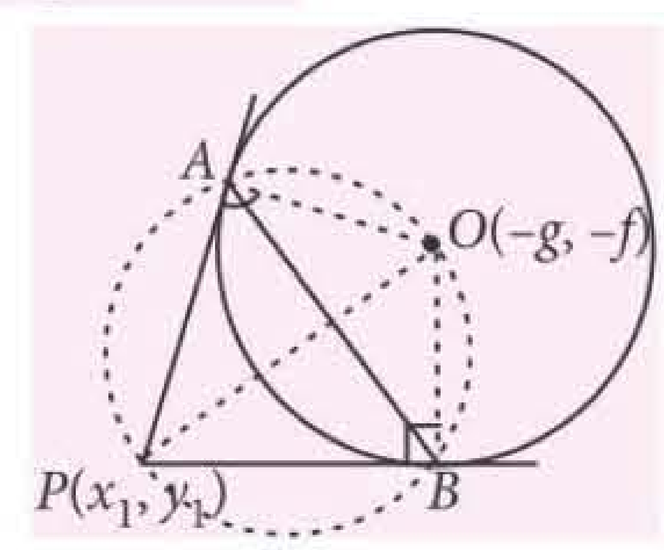
- The point of intersection of the tangents at the points $P(a\cos\alpha, a\sin\alpha)$ and $Q(a\cos\beta, a\sin\beta)$ on the circle $x^2 + y^2 = a^2$ is

$$\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{a\sin\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}\right)$$

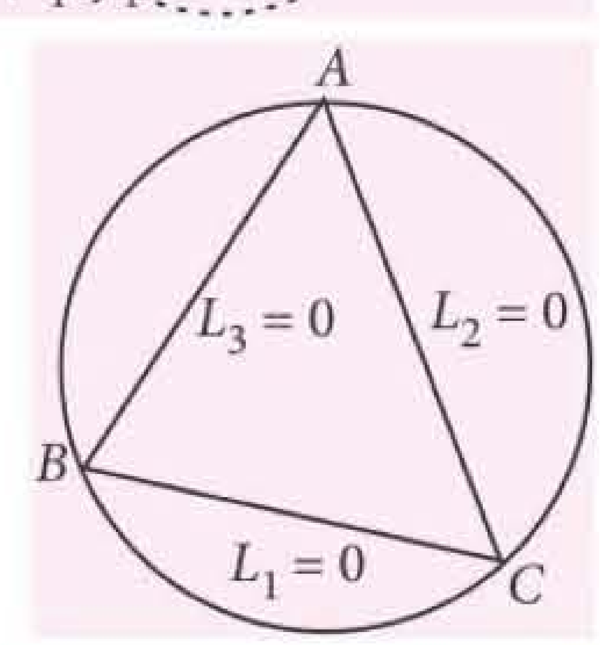
- Length of chord of contact is $AB = \dfrac{2LR}{\sqrt{(R^2 + L^2)}}$ and area of the triangle formed by the pair of tangents and its chord of contact is $\dfrac{RL^2}{R^2 + L^2}$ where R is the radius of the circle and L is the lengths of tangents from $P(x_1, y_1)$ on $S = 0$. Here $L = \sqrt{S_1}$.



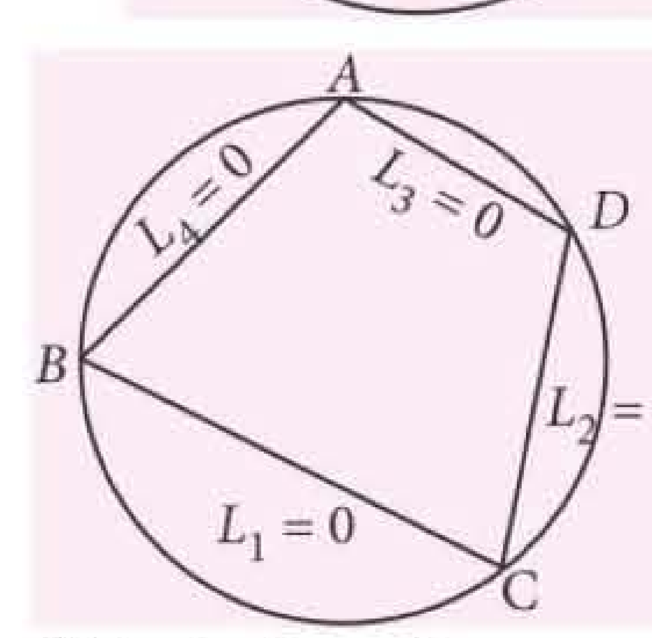
- Equation of the circle circumscribing the triangle PAB is $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$ where $O(-g, -f)$ is the centre of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$



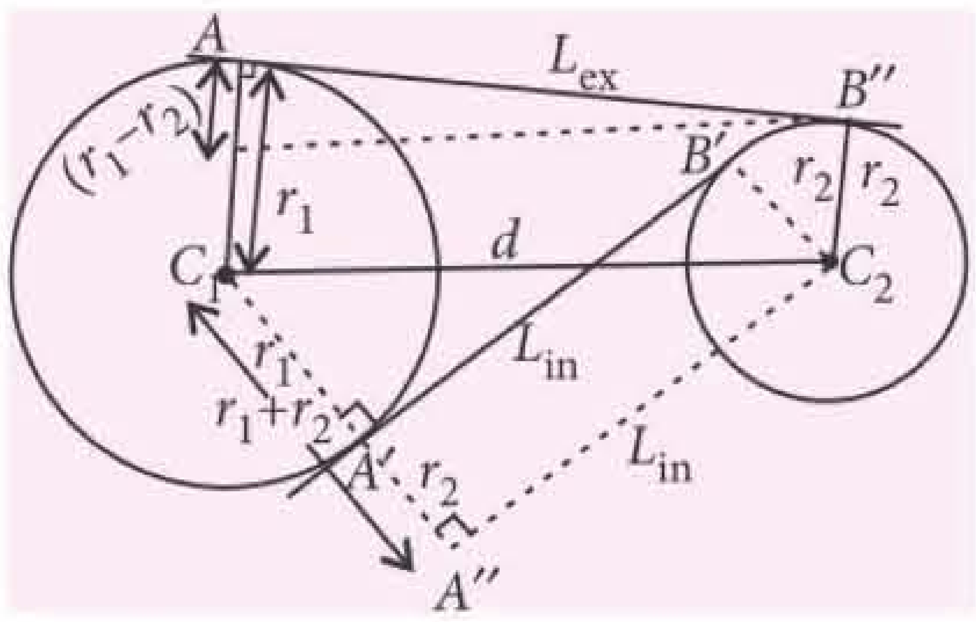
- Family of circles circumscribing a triangle whose sides are given by $L_1 = 0, L_2 = 0$ and $L_3 = 0$ is given by $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ and coefficient of x^2 = coefficient of y^2.



- Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ is given by $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.



- Length of an external common tangent and internal common tangent to two circles is given by the length of external common tangent $L_{ex} = \sqrt{d^2 - (r_1 - r_2)^2}$ and length of internal common tangent $L_{in} = \sqrt{d^2 - (r_1 + r_2)^2}$ [Applicable only when $d > (r_1 + r_2)$] where d is the distance between the centres of circles and r_1 and r_2 are the radii of two circles.



- The locus of the middle point of a chord of a circle subtending a right angle at a given point will be a circle.
- The length of a side of an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$ is $a\sqrt{3}$.
- The distance between the chord of contact of tangents to $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f) is $\dfrac{|g^2+f^2-c|}{2\sqrt{(g^2+f^2)}}$
- The shortest chord of a circle passing through a point P inside the circle is the chord whose middle point is P.
- The length of transverse common tangent < the length of direct common tangent.
- The angle between the two tangents from (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $2\tan^{-1}\left(\dfrac{a}{\sqrt{S_1}}\right)$; where $S_1 = x_1^2 + y_1^2 - a^2$.

PROBLEMS

1. The equation of the circle of radius 5 in the first quadrant which touches x-axis and the line $4y = 3x$ is
(a) $x^2 + y^2 - 24x - y - 25 = 0$
(b) $x^2 + y^2 - 30x - 10y + 225 = 0$
(c) $x^2 + y^2 - 16x - 18y + 64 = 0$
(d) $x^2 + y^2 - 20x - 12y + 144 = 0$

2. Find the equation of the circle which passes through the point of intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$ and whose centre is $(2, -3)$.
(a) $(x-2)^2+(y+3)^2=(\sqrt{109})^2$
(b) $(x+2)^2-(y-3)^2=(\sqrt{109})^2$
(c) $(x-2)^2-(y+3)^2=(\sqrt{109})^2$
(d) $(x-2)^2-(y-3)^2=(\sqrt{109})^2$

3. If θ is the angle between the tangents from $(-1, 0)$ to the circle $x^2 + y^2 - 5x + 4y - 2 = 0$, then θ is equal to
(a) $2\tan^{-1}\left(\dfrac{7}{4}\right)$ (b) $\tan^{-1}\left(\dfrac{7}{4}\right)$
(c) $2\cot^{-1}\left(\dfrac{7}{4}\right)$ (d) $\cot^{-1}\left(\dfrac{7}{4}\right)$

4. The centre of a circle is $(2, -3)$ and the circumference is 10π. Then the equation of the circle is
(a) $x^2 + y^2 + 4x + 6y + 12 = 0$
(b) $x^2 + y^2 - 4x + 6y + 12 = 0$
(c) $x^2 + y^2 - 4x + 6y - 12 = 0$
(d) $x^2 + y^2 - 4x - 6y - 12 = 0$

5. The equation of the circle which passes through the intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ and whose centre lies on $13x + 30y = 0$ is
(a) $x^2 + y^2 + 30x - 13y - 25 = 0$
(b) $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
(c) $2x^2 + 2y^2 + 30x - 13y - 25 = 0$
(d) $x^2 + y^2 + 30x - 13y + 25 = 0$

6. The equation of the circle on the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y + b)^2 = b^2$ as diameter, is
(a) $x^2 + y^2 = 2ab(bx + ay)$ (b) $x^2 + y^2 = bx + ay$
(c) $(a^2 + b^2)(x^2 + y^2) = 2ab(bx - ay)$
(d) $(a^2 + b^2)(x^2 + y^2) = 2(bx + ay)$

7. If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q then the line $5x + by - a = 0$ passes through P and Q for
(a) exactly one value of a (b) no value of a
(c) infinitely many values of a
(d) exactly two values of a

8. To which of the following circles, the line $y - x + 3 = 0$ is normal at the point $\left(3+\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$?
(a) $\left(x-3-\dfrac{3}{\sqrt{2}}\right)^2+\left(y-\dfrac{3}{\sqrt{2}}\right)^2=9$
(b) $\left(x-\dfrac{3}{\sqrt{2}}\right)^2+\left(y-\dfrac{3}{\sqrt{2}}\right)^2=9$
(c) $x^2 + (y - 3)^2 = 9$ (d) $(x - 3)^2 + y^2 = 9$

9. The equation of the circle which touches both the axes in I quadrant and whose radius is a, is
(a) $x^2 + y^2 - 2ax - 2ay + a^2 = 0$
(b) $x^2 + y^2 + ax + ay - a^2 = 0$
(c) $x^2 + y^2 + 2ax + 2ay - a^2 = 0$
(d) $x^2 + y^2 - ax - ay + a^2 = 0$

10. The equation of pair of tangents drawn from the point $(0, 1)$ to the circle $x^2 + y^2 - 2x + 4y = 0$ is
(a) $4x^2 - 4y^2 + 6xy + 6x + 8y - 4 = 0$
(b) $4x^2 - 4y^2 + 6xy - 6x + 8y - 4 = 0$
(c) $x^2 - y^2 + 3xy - 3x + 2y - 1 = 0$
(d) $x^2 - y^2 + 6xy - 6x + 8y - 4 = 0$

11. The equation of the circle which passes through points of intersection of circles $x^2 + y^2 + 4x - 5y + 3 = 0$ and $x^2 + y^2 + 2x + 3y - 3 = 0$ and point $(-3, 2)$ is
(a) $x^2 + y^2 + 8x + 13y - 3 = 0$
(b) $4x^2 + 4y^2 + 13x - 8y + 3 = 0$
(c) $x^2 + y^2 - 13x - 8y + 3 = 0$
(d) $x^2 + y^2 - 13x + 8y + 3 = 0$

12. Tangents are drawn to the circle $x^2 + y^2 = 9$ at the points where it is met by the circle $x^2 + y^2 + 3x + 4y + 2 = 0$. The point of intersection of these tangents will be
(a) $\left(\dfrac{8}{11}, \dfrac{13}{11}\right)$ (b) $\left(-\dfrac{27}{11}, -\dfrac{36}{11}\right)$
(c) $\left(\dfrac{11}{27}, \dfrac{11}{36}\right)$ (d) $\left(\dfrac{3}{4}, 1\right)$

13. Suppose that two circles C_1 and C_2 in a plane have no points in common. Then
(a) there are exactly two line tangent to both C_1 and C_2
(b) there are exactly 3 lines tangent to both C_1 and C_2
(c) there are no lines tangent to both C_1 and C_2 or there are exactly two lines tangent to both C_1 and C_2
(d) there are no lines tangent to both C_1 and C_2 or there are exactly four lines tangent to both C_1 and C_2

14. The equation of the circle which passes through the origin and cuts orthogonally each of the two circles $x^2 + y^2 - 6x + 8 = 0$ and $x^2 + y^2 - 2x - 2y - 7 = 0$ is
(a) $3x^2 + 3y^2 - 8x - 13y = 0$
(b) $3x^2 + 3y^2 - 8x + 29y = 0$
(c) $3x^2 + 3y^2 + 8x + 29y = 0$
(d) $3x^2 + 3y^2 - 8x - 29y = 0$

15. For the two circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 2y = 0$ there is/are
(a) one pair of common tangents
(b) only one common tangent
(c) three common tangents
(d) no common tangent

16. Find the equation of the circle passing through the point (2, 1) and touching the line $x + 2y - 1 = 0$ at the point (3, –1).
(a) $3(x^2 + y^2) - 23x - 4y + 35 = 0$
(b) $x^2 - y^2 + 23x + 4y - 35 = 0$
(c) $2x^2 - 2y^2 - 23x - 4y + 35 = 0$
(d) None of these

17. If equation $x^2 + y^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle, then the condition for that circle to pass, through three quadrants only but not passing through the origin is
(a) $f^2 > c, g^2 > c, c > 0$
(b) $g^2 > c, f^2 < c, c > 0, h = 0$
(c) $f^2 > c, g^2 > c, c > 0, h = 0$
(d) $g^2 < c, f^2 < c, c < 0, h = 0$

18. The equation of the circle which has a tangent $2x - y - 1 = 0$ at (3, 5) on it and with the centre on $x + y = 5$, is
(a) $x^2 + y^2 + 6x - 16y + 28 = 0$
(b) $x^2 + y^2 - 6x - 16y - 28 = 0$
(c) $x^2 + y^2 + 6x + 6y - 28 = 0$
(d) $x^2 + y^2 - 6x - 6y - 28 = 0$

19. The distance from the centre of the circle $x^2 + y^2 = 2x$ to straight line passing through the points of intersection of the two circles $x^2 + y^2 + 5x - 8y + 1 = 0$ and $x^2 + y^2 - 3x + 7y - 25 = 0$ is
(a) 1/3 (b) 2 (c) 3 (d) 1

20. Two circles with radii r_1 and $r_2 (r_1 > r_2 \geq 2)$ touch each other externally. If θ be the angle between the direct common tangents, then
(a) $\theta = \sin^{-1}\left(\frac{r_1 + r_2}{r_1 - r_2}\right)$ (b) $\theta = 2\sin^{-1}\left(\frac{r_1 - r_2}{r_1 + r_2}\right)$
(c) $\theta = \sin^{-1}\left(\frac{r_1 - r_2}{r_1 + r_2}\right)$ (d) None of these

21. If the curves $ax^2 + 4xy + 2y^2 + x + y + 5 = 0$ and $ax^2 + 6xy + 5y^2 + 2x + 3y + 8 = 0$ intersect at four concyclic points then the value of a is
(a) 4 (b) –4 (c) 6 (d) –6

22. Two perpendicular tangents to the circle $x^2 + y^2 = a^2$ meet at P. Then, the locus of P has the equation
(a) $x^2 + y^2 = 2a^2$ (b) $x^2 + y^2 = 3a^2$
(c) $x^2 + y^2 = 4a^2$ (d) None of these

23. The area of the triangle formed by the tangents from an external point (h, k) to the circle $x^2 + y^2 = a^2$ and the chord of contact, is
(a) $\frac{1}{2}a\left(\frac{h^2 + k^2 - a^2}{\sqrt{h^2 + k^2}}\right)$ (b) $\frac{a(h^2 + k^2 - a^2)^{3/2}}{2(h^2 + k^2)}$
(c) $\frac{a(h^2 + k^2 - a^2)^{3/2}}{(h^2 + k^2)}$ (d) None of these

24. The locus of a point which moves so that the ratio of the length of the tangents to the circles $x^2 + y^2 + 4x + 3 = 0$ and $x^2 + y^2 - 6x + 5 = 0$ is 2 : 3, is
(a) $5x^2 + 5y^2 + 60x - 7 = 0$ (b) $5x^2 + 5y^2 - 60x - 7 = 0$
(c) $5x^2 + 5y^2 + 60x + 7 = 0$ (d) $5x^2 + 5y^2 + 60x + 12 = 0$

25. The circle S_1 with centre $C_1(a_1, b_1)$ and radius r_1 touches externally the circle S_2 with centre $C_2(a_2, b_2)$ and radius r_2. If the tangent at their common point passes through the origin, then
(a) $(a_1^2 + a_2^2) + (b_1^2 + b_2^2) = r_1^2 + r_2^2$
(b) $(a_2^2 - a_1^2) + (b_2^2 - b_1^2) = r_2^2 - r_1^2$
(c) $(a_1^2 - b_2^2) + (a_2^2 + b_2^2) = r_1^2 + r_2^2$
(d) $(a_1^2 - b_1^2) + (a_2^2 + b_2^2) = r_{12} + r_2^2$

26. If two circles, each of radius 5 unit, touch each other at (1, 2) and the equation of their common tangent is $4x + 3y = 10$, then equation of the circle a portion of which lies in all the quadrants, is
(a) $x^2 + y^2 - 10x - 10y + 25 = 0$
(b) $x^2 + y^2 + 6x + 2y - 15 = 0$
(c) $x^2 + y^2 + 2x + 6y - 15 = 0$
(d) $x^2 + y^2 + 10x + 10y + 25 = 0$

27. A rhombus is inscribed in the region common to the two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centres of the circles. The area of the rhombus is
(a) $8\sqrt{3}$ sq. units (b) $4\sqrt{3}$ sq. units
(c) $16\sqrt{3}$ sq. units (d) None of these

28. The equations of three circles are given:
$x^2 + y^2 = 1$, $x^2 + y^2 - 8x + 15 = 0$, $x^2 + y^2 + 10y + 24 = 0$.
The coordinates of the point such that the tangents drawn from it to three circles are equal in length, are
(a) $\left(2, \frac{5}{2}\right)$ (b) $\left(2, \frac{-5}{2}\right)$

(c) $\left(-2, \frac{5}{2}\right)$ (d) $\left(-2, \frac{-5}{2}\right)$

29. If the tangents are drawn from any point on the line $x + y = 3$ to the circle $x^2 + y^2 = 9$, then the chord of contact passes through the point

(a) (3, 5) (b) (3, 3)
(c) (5, 3) (d) None of these

30. The slope of the tangent at the point (h, h) on the circle $x^2 + y^2 = a^2$ is

(a) 0 (b) 1
(c) −1 (d) dependent of h

SOLUTIONS

1. (b) : Let the centre of circle be $(g, 5)$.

$\therefore \frac{3(g)-4(5)}{\sqrt{3^2+4^2}} = 5 \Rightarrow 3g = 25 + 20 \Rightarrow g = 15$

$\therefore$ Equation of circle whose centre is (15, 5) and radius 5 is $(x - 15)^2 + (y - 5)^2 = 5^2$

$\Rightarrow x^2 - 30x + y^2 - 10y + 225 = 0$

2. (a) : Let P be the point of intersection of the lines AB and LM whose equations are respectively

$3x - 2y - 1 = 0$ ...(i) and $4x + y - 27 = 0$ .... (ii)

Solving (i) and (ii), we get $x = 5$, $y = 7$. So, coordinates of P are (5, 7). It is given that $C(2, -3)$ be the centre of the circle. Since the circle passes through P, therefore

$CP = \text{radius} = \sqrt{(5-2)^2+(7+3)^2} \Rightarrow \text{radius} = \sqrt{109}$

Hence the equation of the required circle is

$(x-2)^2+(y+3)^2 = (\sqrt{109})^2$

3. (a) : We know that, the angle between the two tangents from (α, β) to the circle $x^2 + y^2 = r^2$ is

$2\tan^{-1}\frac{r}{\sqrt{S_1}}$

Let $S = x^2 + y^2 - 5x + 4y - 2$

Here, $r = \sqrt{\left(-\frac{5}{2}\right)^2 + (2)^2 + 2} = \frac{7}{2}$

At point $(-1, 0)$, $S_1 = (-1)^2 + (0)^2 - 5(-1) + 4(0) - 2 = 4$

$\therefore$ Required angle, $\theta = 2\tan^{-1}\frac{7/2}{\sqrt{4}} = 2\tan^{-1}\left(\frac{7}{4}\right)$

4. (c) : It is given, centre is $(2, -3)$ and circumference of circle $= 10\pi \Rightarrow 2\pi r = 10\pi \Rightarrow r = 5$

$\therefore$ The equation of circle is $(x - 2)^2 + (y + 3)^2 = 5^2$

$\Rightarrow x^2 + y^2 - 4x + 6y + 13 = 25$

$\Rightarrow x^2 + y^2 - 4x + 6y - 12 = 0$

5. (b) : Let the equation of circles be

$S_1 \equiv x^2 + y^2 + 13x - 3y = 0$ ... (i)

and $S_2 \equiv 2x^2 + 2y^2 + 4x - 7y - 25 = 0$ ... (ii)

The equation of intersecting circle is $\lambda S_1 + S_2 = 0$

$\Rightarrow \lambda(x^2+y^2+13x-3y)+\left(x^2+y^2+2x-\frac{7y}{2}-\frac{25}{2}\right)=0$ ...(iii)

$\therefore$ Centre $= \left(-\frac{(2+13\lambda)}{2(1+\lambda)}, \frac{(7/2)+3\lambda}{2(1+\lambda)}\right)$

$\because$ Centre lies on $13x + 30y = 0$.

$\therefore -13\left(\frac{2+13\lambda}{2}\right)+30\left(\frac{(7/2)+3\lambda}{2}\right)=0$

$\Rightarrow -26 - 169\lambda + 105 + 90\lambda = 0 \Rightarrow \lambda = 1$

Hence, putting the value of λ in (iii), we get required equation of circle as $4x^2 + 4y^2 + 30x - 13y - 25 = 0$

6. (c) : The equation of the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y + b)^2 = b^2$ is

$I \equiv S_1 - S_2 = 0$

$\Rightarrow x^2 + a^2 - 2ax + y^2 - a^2 - x^2 - y^2 - b^2 - 2by + b^2 = 0$

$\Rightarrow ax + by = 0$ ... (i)

Now, the equation of required circle is $S_1 + \lambda L = 0$

$\therefore \{(x - a)^2 + y^2 - a^2\} + \lambda\{ax + by\} = 0$

$\Rightarrow x^2 + y^2 + x(a\lambda - 2a) + \lambda by = 0$ ... (ii)

Since, (i) is a diameter of (ii).

$\therefore a\left(-\frac{a\lambda-2a}{2}\right)+b\left(-\frac{\lambda b}{2}\right)=0 \Rightarrow \lambda = \frac{2a^2}{a^2+b^2}$

On putting the value of λ in (ii), we get

$(a^2 + b^2)(x^2 + y^2) = 2ab(bx - ay)$

which is the required equation of circle.

7. (b) : $S_1 - S_2 = 5ax + (c - d)y + a + 1 = 0$ and $5x + by - a = 0$ must represent the same line.

$\therefore \frac{a}{1} = \frac{c-d}{b} = \frac{a+1}{-a}$

$\Rightarrow ab = c - d$ and $a^2 + a + 1 = 0$

Thus, a is imaginary so no value of a exists.

8. (d) : Line must pass through the centre of the circle.

9. (a) : Required equation is $(x - a)^2 + (y - a)^2 = a^2$

$\Rightarrow x^2 + y^2 - 2ax - 2ay + a^2 = 0$

10. (b) : Let $S = x^2 + y^2 - 2x + 4y$ then

$S_1 = 0^2 + 1^2 - 2\cdot 0 + 4\cdot 1 = 5$

$T = x\cdot 0 + y\cdot 1 - (x+0) + 2(y+1) = -x + 3y + 2$

The equation of the pair of tangents is $SS_1 = T^2$

$\Rightarrow (x^2 + y^2 - 2x + 4y)5 = (-x + 3y + 2)^2$

$\Rightarrow 4x^2 - 4y^2 + 6xy - 6x + 8y - 4 = 0$

11. (b) : The equation of circle through the points of intersection of given circles is

$x^2 + y^2 + 4x - 5y + 3 + \lambda(x^2 + y^2 + 2x + 3y - 3) = 0$

Since it passes through point $(-3, 2)$ also, therefore

$-6 + 10\lambda = 0 \Rightarrow \lambda = \frac{3}{5}$

Hence equation of required circle is

$5x^2 + 5y^2 + 20x - 25y + 15 + 3x^2 + 3y^2 + 6x + 9y - 9 = 0$

$\Rightarrow 8x^2 + 8y^2 + 26x - 16y + 6 = 0$

$\Rightarrow 4x^2 + 4y^2 + 13x - 8y + 3 = 0$

12. (b) : Equation of common chord will be

$3x + 4y + 11 = 0$ ...(i)

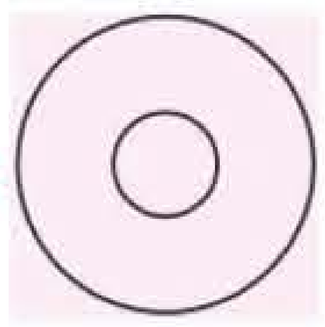
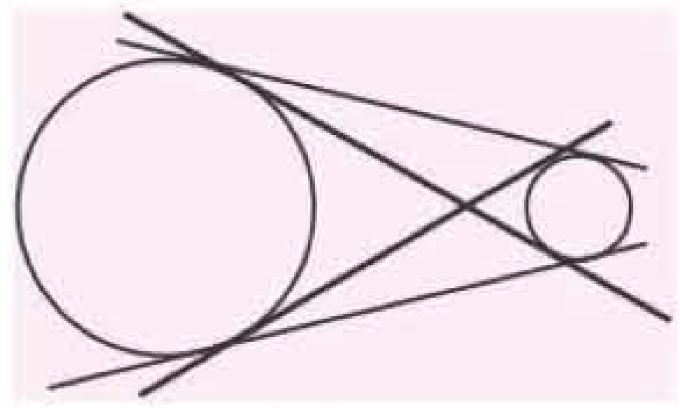
Let the point of intersection of the tangents be (α, β).

$\therefore$ Equation of the chord of contact of the tangents drawn from (α, β) to first circle will be

$x\alpha + y\beta = 9$ ...(ii)

Since, (i) and (ii) are identical.

$\therefore \quad \frac{3}{\alpha} = \frac{4}{\beta} = -\frac{11}{9} \Rightarrow (\alpha, \beta) = \left(\frac{-27}{11}, -\frac{36}{11}\right)$

13. (d) :  or 

14. (b) : Let required equation of circle be

$x^2 + y^2 + 2gx + 2fy = 0$

Since, the above circle cuts the given circles orthogonally.

$\therefore \quad 2(-3g) + 2f(0) = 8 \Rightarrow 2g = -8/3$

and $-2g - 2f = -7 \Rightarrow 2f = 7 + \frac{8}{3} = \frac{29}{3}$

$\therefore$ Required equation of the circle is

$x^2 + y^2 - \frac{8}{3}x + \frac{29}{3}y = 0 \Rightarrow 3x^2 + 3y^2 - 8x + 29y = 0$

15. (d) : The centres and radii of given circles are

$C_1(0, 0),\ r_1 = 4,\ C_2(0, 1),\ r_2 = \sqrt{0+1} = 1$

Now, $C_1C_2 = \sqrt{0 + (0-1)^2} = 1$ and $r_1 - r_2 = 3$.

$\therefore \quad C_1C_2 < r_1 - r_2$

Hence, second circle lies inside the first circle, so no common tangent is possible.

16. (a) : Equation of circle is

$(x - 3)^2 + (y + 1)^2 + \lambda(x + 2y - 1) = 0$

Since, it passes through the point (2, 1).

$\therefore \quad 1 + 4 + \lambda(2 + 2 - 1) = 0 \Rightarrow \lambda = -\frac{5}{3}$

$\therefore$ Circle is $(x-3)^2 + (y+1)^2 - \frac{5}{3}(x + 2y - 1) = 0$

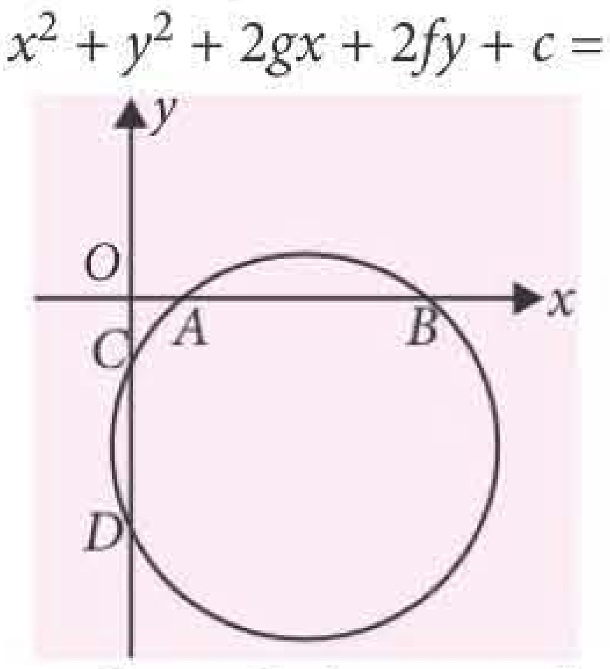
$\Rightarrow \quad 3x^2 + 3y^2 - 23x - 4y + 35 = 0$

17. (c) : Given circle is

$x^2 + y^2 + 2hxy + 2gx + 2fy + c = 0$ ...(i)

For (i) to represent a circle, $h = 0$

So, given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(ii)



For circle (ii) to pass through three quadrants only.

(I) $AB > 0 \Rightarrow g^2 - c > 0$

(II) $CD > 0 \Rightarrow f^2 - c > 0$

(III) Origin should be outside circle (ii).

$\therefore \quad c > 0$

From (I), (II) and (III), $g^2 > c, f^2 > c, c > 0$

$\therefore$ Required conditions are

$g^2 > c, f^2 > c, c > 0, h = 0$

18. (a) : Clearly, the centre of the circle lies on the line through the point (3, 5) perpendicular to the tangent $2x - y - 1 = 0$.

The equation of such line is

$(y - 5) = \frac{-1}{2}(x - 3) \Rightarrow x + 2y = 13$ ...(i)

Also, it is given that centre lies on the line

$x + y = 5$ ...(ii)

Solving (i) and (ii), we obtain the coordinates of the centre of circle as $C \equiv (-3, 8)$

Also, radius of the circle $= \sqrt{36 + 9} = \sqrt{45}$

$\therefore$ Equation of the circle is

$(x + 3)^2 + (y - 8)^2 = (\sqrt{45})^2$

$\Rightarrow \quad x^2 + y^2 + 6x - 16y + 28 = 0$

19. (b) : The equation of the straight line passing through the points of intersection of given circles is

$(x^2 + y^2 + 5x - 8y + 1) - (x^2 + y^2 - 3x + 7y - 25) = 0$

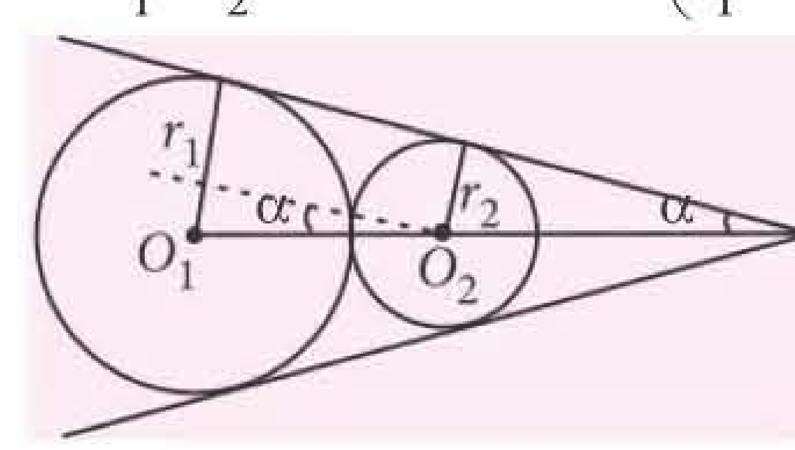
$\Rightarrow \quad 8x - 15y + 26 = 0$ ...(i)

Also, centre of the circle $x^2 + y^2 - 2x = 0$ is (1, 0).

$\therefore$ Distance of the point (1, 0) from the straight line (i) is given by

$$d = \frac{|8(1) - 15(0) + 26|}{\sqrt{64 + 225}} = \frac{34}{17} = 2$$

20. (b) : $\sin\alpha = \frac{r_1 - r_2}{r_1 + r_2} \Rightarrow \theta = 2\sin^{-1}\left(\frac{r_1 - r_2}{r_1 + r_2}\right)$



21. (b) : Any second degree curve passing through the intersection of the given curves is

$ax^2 + 4xy + 2y^2 + x + y + 5 + \lambda \times (ax^2 + 6xy + 5y^2 + 2x + 3y + 8) = 0$

If it is a circle, then coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$

$a(1 + \lambda) = 2 + 5\lambda$ and $4 + 6\lambda = 0$

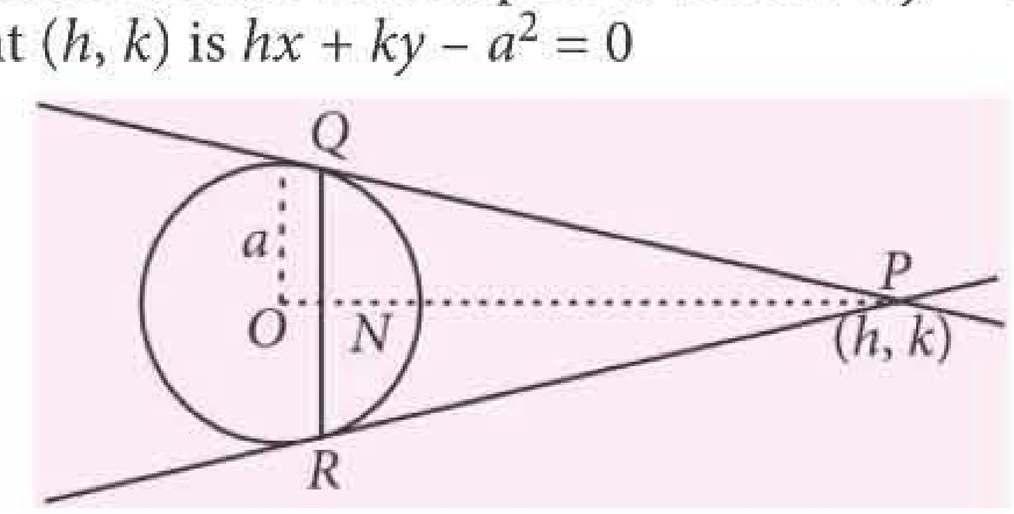
$\Rightarrow \quad a = \frac{2 + 5\lambda}{1 + \lambda}$ and $\lambda = -\frac{2}{3} \Rightarrow a = \frac{2 - \frac{10}{3}}{1 - \frac{2}{3}} = -4$

22. (a) : We know that, if two perpendicular tangents to the circle $x^2 + y^2 = a^2$ meet at P, then the point P lies on a director circle. Thus, the equation of director circle to the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$

which is the required locus of point P.

23. (c) : Here, area of ΔPQR is required.

Now chord of contact with respect to circle $x^2 + y^2 = a^2$, and point (h, k) is $hx + ky - a^2 = 0$



Now, length of $\perp r$, $PN = \dfrac{h^2 + k^2 - a^2}{\sqrt{h^2 + k^2}}$

Also, $QR = 2\sqrt{a^2 - \dfrac{(a^2)^2}{h^2 + k^2}} = \dfrac{2a\sqrt{h^2 + k^2 - a^2}}{\sqrt{h^2 + k^2}}$

$\therefore$ Area of $\Delta PQR = \dfrac{1}{2}(QR)(PN)$

$$= \frac{1}{2} 2a \frac{\sqrt{h^2 + k^2 - a^2}}{\sqrt{h^2 + k^2}} \times \frac{(h^2 + k^2 - a^2)}{\sqrt{h^2 + k^2}}$$

$$= a\frac{(h^2 + k^2 - a^2)^{3/2}}{h^2 + k^2}$$

24. (c) : Let $P(x_1, y_1)$ be any point outside the circle. Length of tangent to the circle $x^2 + y^2 + 4x + 3 = 0$ is $\sqrt{x_1^2 + y_1^2 + 4x_1 + 3}$ and length of tangent to the circle $x^2 + y^2 - 6x + 5 = 0$ is

$\sqrt{x_1^2 + y_1^2 - 6x_1 + 5}$

According to question, $\dfrac{\sqrt{x_1^2 + y_1^2 + 4x_1 + 3}}{\sqrt{x_1^2 + y_1^2 - 6x_1 + 5}} = \dfrac{2}{3}$

$\Rightarrow \quad 9x_1^2 + 9y_1^2 + 36x_1 + 27 - 4x_1^2 - 4y_1^2 + 24x_1 - 20 = 0$

$\Rightarrow \quad 5x_1^2 + 5y_1^2 + 60x_1 + 7 = 0$

$\therefore \quad$ Locus of point P is $5x^2 + 5y^2 + 60x + 7 = 0$.

25. (b) : The two circles are

$S_1 = (x - a_1)^2 + (y - b_1)^2 = r_1^2 \quad$ ... (i)

$S_2 = (x - a_2)^2 + (y - b_2)^2 = r_2^2 \quad$ ... (ii)

The equation of the common tangent of these two circles is given by $S_1 - S_2 = 0$

$\Rightarrow \quad 2x(a_1 - a_2) + 2y(b_1 - b_2) + (a_2^2 + b_2^2) - (a_1^2 + b_1^2) + r_1^2 - r_2^2 = 0$

If this passes through the origin, then

$(a_2^2 + b_2^2) - (a_1^2 + b_1^2) + r_1^2 - r_2^2 = 0$

$\Rightarrow \quad (a_2^2 - a_1^2) + (b_2^2 - b_1^2) = r_2^2 - r_1^2$

26. (b) : The centres of the two circles will lie on the line through $P(1, 2)$ and perpendicular to the common tangent $4x + 3y = 10$. If C_1 and C_2 are the centres of these circles, then $PC_1 = 5 = r_1$ and $PC_2 = 5 = r_2$.

Also, C_1, C_2 lie on the line $\dfrac{x-1}{\cos\theta} = \dfrac{y-2}{\sin\theta} = r$, where $\tan\theta = \dfrac{3}{4}$.

When $r = r_1$, the coordinates of C_1 are

$(5\cos\theta + 1, 5\sin\theta + 2)$ or $(5, 5)$ as $\cos\theta = \dfrac{4}{5}$, $\sin\theta = \dfrac{3}{5}$

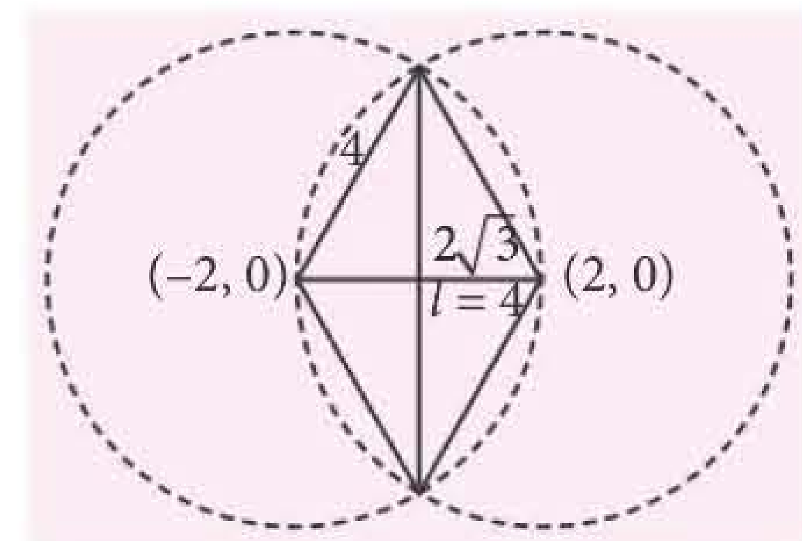
When $r = r_2$, the coordinates of C_2 are $(-3, -1)$.

The circle with centre $C_1(5, 5)$ and radius 5 touches both the coordinates axes and hence lies completely in the first quadrant. Therefore, the required circle has centre $(-3, -1)$ and radius 5, so its equation is

$(x + 3)^2 + (y + 1)^2 = 5^2 \Rightarrow x^2 + y^2 + 6x + 2y - 15 = 0$

Since, the origin lies inside the circle, a portion of the circle lies in all the quadrants.

27. (a) : We have, circles with centre $(2, 0)$ and $(-2, 0)$ each with radius 4. So, y-axis is their common chord.



The inscribed rhombus has its diagonals equal to 4 and $4\sqrt{3}$.

$\therefore \quad$ Area of rhombus $= \dfrac{d_1 d_2}{2} = 8\sqrt{3}$

28. (b) : Let (x_1, y_1) be the point. As the tangents from (x_1, y_1) to the first two circles are equal, (x_1, y_1) is on the radical axis of the circles, its equation being

$S_1 - S_2 = (x^2 + y^2 - 1) - (x^2 + y^2 - 8x + 15) = 0$

$\Rightarrow \quad 8x - 16 = 0 \Rightarrow x - 2 = 0 \quad$ ...(i)

Similarly, (x_1, y_1) is on the radical axis of the second and third circle whose equation is

$S_2 - S_3 = x^2 + y^2 - 8x + 15 - (x^2 + y^2 + 10y + 24) = 0$

$\Rightarrow \quad 8x + 10y + 9 = 0 \quad$ ...(ii)

Solving (i) and (ii), we get $x = 2$ and $y = -5/2$

$\therefore \quad$ The required point is $\left(2, -\dfrac{5}{2}\right)$.

29. (b) : The coordinates of any point on the line $x + y = 3$ are $(k, 3 - k)$. The equation of chord of contact of tangents drawn from $(k, 3 - k)$ to the circle $x^2 + y^2 = 9$ is $kx + (3 - k) \cdot y = 9$

$\Rightarrow \quad (3y - 9) + k(x - y) = 0$

which clearly passes through the intersection of

$3y - 9 = 0$ and $x - y = 0$ *i.e.*, $(3, 3)$.

30. (c) : The equation of the tangent at (h, h) to $x^2 + y^2 = a^2$ is $hx + hy = a^2$.

Therefore, slope of the tangent $= -h/h = -1$

MOCK TEST PAPER 2020

JEE MAIN

Exam date: 18^{th} to 23^{rd} July, 2020

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1. If $2\tan^2 x - 5\sec x = 1$ for exactly 7 distinct values of $x \in \left[0, \frac{n\pi}{2}\right], n \in N$ then the greatest value of n is

(a) 13 (b) 17 (c) 19 (d) 15

2. Let $\theta \in [0, 4\pi]$ satisfying the equation $(\sin\theta + 2)(\sin\theta + 3)(\sin\theta + 4) = 6$. If the sum of all values of θ is $K\pi$ then value of K is

(a) 6 (b) 5 (c) 4 (d) 2

3. The number of solutions of the equation $16(\sin^5 x + \cos^5 x) = 11(\sin x + \cos x)$ in the interval $[0, 2\pi]$ is

(a) 6 (b) 7 (c) 8 (d) 9

4. The equation $2x = (2n + 1)\pi(1 - \cos x)$, (where n is a positive integer)

(a) has infinitely many real roots
(b) has exactly one real root
(c) has exactly $2n + 2$ real roots
(d) has exactly $2n + 3$ real roots

5. Number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is

(a) 0 (b) 1 (c) 2 (d) 3

6. $|x^2 \sin x + \cos^2 x e^x + \ln^2 x| < x^2|\sin x| + \cos^2 x e^x + \ln^2 x$ true for $x \in$

(a) $(-\pi, 0)$ (b) $\left(0, \frac{\pi}{2}\right)$

(c) $\left(\frac{\pi}{2}, \pi\right)$ (d) $(2n\pi, (2n + 1)\pi)\ n \in N$

7. If $\log_{0.5}\sin x = 1 - \log_{0.5}\cos x$ then number of values of $x \in [-2\pi, 2\pi]$ is

(a) 1 (b) 2 (c) 3 (d) 4

8. If $2^{\sqrt{\sin^2 x - 2\sin x + 5}}\dfrac{1}{4^{\sin^2 y}} \le 1$, then the ordered pair (x, y) is equal to $(m, n \in I)$

(a) $x = (4n+1)\frac{\pi}{2}, y = (2m+1)\frac{\pi}{2}$
(b) $x = 2n\pi, y = 2m\pi$
(c) $x = (2n+1)\frac{\pi}{2}, y = (2m+1)\frac{\pi}{2}$
(d) $x = n\pi, y = m\pi$

9. The value of $\cos^2 10° - \cos 10° \cos 50° + \cos^2 50° =$

(a) $\frac{4}{5}$ (b) $\frac{1}{3}$ (c) $\frac{3}{4}$ (d) 3

10. Number of ordered pairs (a, x) satisfying the equation $\sec^2(a + 2)x + a^2 - 1 = 0; -\pi < x < \pi$ is

(a) 1 (b) 2 (c) 3 (d) 5

11. If $a\sin^2 x + b\cos^2 x = c$, $b\sin^2 y + a\cos^2 y = d$ and $a\tan x = b\tan y$ then $\frac{a^2}{b^2} =$

(a) $\frac{(a-d)(c-a)}{(b-c)(d-b)}$ (b) $\frac{(a+d)(c+a)}{(b+c)(d+b)}$

(c) $\frac{(a-d)(b-a)}{(a-c)(c-b)}$ (d) $\frac{(d-a)(c-a)}{(b-c)(d-b)}$

12. If $\cos\theta = \frac{a\cos\phi + b}{a + b\cos\phi}$ then $\tan\frac{\theta}{2}$ is equal to

(a) $\sqrt{\left(\frac{a-b}{a+b}\right)}\tan(\phi/2)$ (b) $\sqrt{\left(\frac{a+b}{a-b}\right)}\cos(\phi/2)$

(c) $\sqrt{\left(\frac{a-b}{a+b}\right)}\sin(\phi/2)$ (d) none of these

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13. If α is the angle in which each side of a regular polygon of n sides subtends at its centre then $1 + \cos\alpha + \cos2\alpha + \cos3\alpha + \ldots\ldots + \cos(n-1)\alpha$ is equal to

(a) n (b) 0 (c) 1 (d) $n - 1$

14. If in a triangle $\dfrac{s-a}{11}=\dfrac{s-b}{12}=\dfrac{s-c}{13}$ and $\lambda\tan^2(A/2) = 455$, then λ must be

(a) 1155 (b) 1551 (c) 5511 (d) 1515

15. If $2\sin x - \cos 2x = 1$, then $\cos^2 x + \cos^4 x$ is equal to

(a) 1 (b) -1 (c) $-\sqrt{5}$ (d) $\sqrt{5}$

16. A set of values of x, satisfying the equation $\cos^2\left(\dfrac{1}{2}px\right)+\cos^2\left(\dfrac{1}{2}qx\right)=1$ form an arithmetic progression with common difference

(a) $\dfrac{2}{p+q}$ (b) $\dfrac{2}{p-q}$

(c) $\dfrac{\pi}{p+q}$ (d) none of these

17. If $\cos^6\alpha + \sin^6\alpha + k\sin^2 2\alpha = 1\ \forall\ \alpha \in (0, \pi/2)$, then k is

(a) $\dfrac{3}{4}$ (b) $\dfrac{1}{4}$ (c) $\dfrac{1}{3}$ (d) $\dfrac{1}{8}$

18. The most general solution of the equations $\tan\theta = -1$, $\cos\theta = \dfrac{1}{\sqrt{2}}$ is

(a) $n\pi + 7\pi/4$ (b) $n\pi + (-1)^n\dfrac{7\pi}{4}$

(c) $2n\pi + \dfrac{7\pi}{4}$ (d) none of these

19. The least positive values of x satisfying the equation $8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\ldots\infty} = 4^3$ will be (where $|\cos x| < 1$)

(a) $\pi/3$ (b) $2\pi/3$

(c) $\pi/4$ (d) none of these

20. For each real number x such that $-1 < x < 1$, let $A(x)$ be the matrix $(1-x)^{-1}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}$ and $z = \dfrac{x+y}{1+xy}$. Then,

(a) $A(z) = A(x) + A(y)$ (b) $A(z) = A(x)\,[A(y)]^{-1}$

(c) $A(z) = A(x)\,A(y)$ (d) $A(z) = A(x) - A(y)$

21. If A is a square matrix of order 3 such that $|A| = 2$ then $|(\text{adj } A^{-1})^{-1}|$ is

(a) 1 (b) 2 (c) 4 (d) 8

22. If A and B are square matrices of the same order and A is non-singular, then for a positive integer n, $(A^{-1}BA)^n$ is equal to

(a) $A^{-n}B^nA^n$ (b) $A^nB^nA^{-n}$

(c) $A^{-1}B^nA$ (d) $n(A^{-1}BA)$

23. If $A=\begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$ and $B=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then A^8 equals

(a) $4B$ (b) $128B$ (c) $-128B$ (d) $-64B$

24. If $p + q + r = 0$ and $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$, then $k =$

(a) 0 (b) abc (c) pqr (d) $a+b+c$

25. A square matrix P satisfies $P^2 = I - P$, where I is an identity matrix of order as order of P. If $P^n = 5I - 8P$, then $n =$

(a) 4 (b) 5 (c) 6 (d) 7

26. The digits A, B, C are such that the three digit numbers $A88$, $6B8$, $86C$ are divisible by 72, then the determinant $\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$ is divisible by

(a) 76 (b) 144 (c) 216 (d) 276

27. Let A, B be square matrix such that $AB = O$ and B is non singular then

(a) $|A|$ must be zero but A may non zero

(b) A must be zero matrix

(c) nothing can be said in general about A

(d) none of these

28. Let $x > 0$, $y > 0$, $z > 0$ are respectively the 2nd, 3rd, 4th terms of a G.P and

$$\Delta = \begin{vmatrix} x^k & x^{k+1} & x^{k+2} \\ y^k & y^{k+1} & y^{k+2} \\ z^k & z^{k+1} & z^{k+2} \end{vmatrix} = (r-1)^2\left(1-\frac{1}{r^2}\right)$$

(where r is the common ratio) then

(a) $k = -1$ (b) $k = 1$

(c) $k = 0$ (d) None of these

29. $A = [a_{ij}]_{m\times n}$ and $a_{ij} = i^2 - j^2$ then A is necessarily

(a) a unit matrix

(b) symmetric matrix

(c) skew symmetric matrix

(d) zero matrix

30. If $A=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ then $A^{16}=$

(a) $\begin{bmatrix}0 & 256\\ 256 & 0\end{bmatrix}$ (b) $\begin{bmatrix}256 & 0\\ 0 & 256\end{bmatrix}$

(c) $\begin{bmatrix}-16 & 0\\ 0 & -16\end{bmatrix}$ (d) $\begin{bmatrix}0 & 16\\ 16 & 0\end{bmatrix}$

31. If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}, B=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}, a\,b\in N$, then number of matrix 'B' such that $AB = BA$ are

(a) 0 (b) 1
(c) finitely many (d) infinite

32. Let A and B are two non-singular square matrices, and A^T and B^T are the transpose matrices of A and B respectively, then which of the following is correct?

(a) B^TAB is symmetric matrix if and only if A is symmetric
(b) B^TAB is symmetric matrix if and only if B is symmetric
(c) B^TAB is skew symmetric matrix for every matrix A
(d) B^TAB is skew symmetric matrix if B is skew symmetric

33. $|A_{3\times 3}| = 3$, $|B_{3\times 3}| = -1$, and $|C_{2\times 2}| = 2$, then $|2ABC| =$

(a) $2^3(6)$ (b) $2^3(-6)$
(c) $2(-6)$ (d) none of these

34. If A and B are two matrices such that $AB = B$ and $BA = A$, then

(a) $(A^6 - B^5)^3 = A - B$ (b) $(A^5 - B^5)^3 = A^3 - B^3$
(c) $A - B$ is idempotent (d) $A - B$ is nilpotent

35. Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ are two matrices such that $AB = BA$ and $c \neq 0$, then value of $\dfrac{a-d}{3b-c}$ is:

(a) 0 (b) 2 (c) -2 (d) -1

36. Let $f(\alpha)=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$, then $(f(\alpha))^{-1}$ is equal to

(a) $f(\alpha)$ (b) $f(-\alpha)$
(c) $f(\alpha - 1)$ (d) none of these

37. If $A=\begin{bmatrix}5 & -6\\ 1 & -1\end{bmatrix}$ then determinant of $A^{1003} - 5A^{1002}$ is

(a) 1 (b) 2 (c) 4 (d) 6

38. $f: R \to R, f(x) = x|x|$ is

(a) one-one but not onto
(b) onto but not one-one
(c) Both one-one and onto
(d) neither one-one nor onto

39. The domain of the function

$f(x)=\sin^{-1}\left(\log_2\left(\dfrac{x^2}{2}\right)\right)$ is

(a) $[-2, 2]$ (b) $[-2, -1]$
(c) $[1, 2]$ (d) $[-2, -1] \cup [1, 2]$

40. The range of $f(x)=\dfrac{3}{5+4\sin 3x}$ is

(a) $\left[\dfrac{1}{3}, 3\right]$ (b) $\left[\dfrac{1}{3}, 1\right]$

(c) $[1, 3]$ (d) $\left(-\infty, \dfrac{1}{3}\right) \cup (3, \infty)$

41. The range of $\dfrac{x^2-x+1}{x^2+x+1}$ is

(a) $\left[\dfrac{1}{3}, 3\right]$ (b) $\left[\dfrac{1}{3}, 1\right]$

(c) $[1, 3]$ (d) $(-\infty, \dfrac{1}{3}] \cup [3, \infty)$

42. Domain of the function $f(x)=\sqrt{5|x|-x^2-6}$ is

(a) $(-\infty, 2) \cup (3, \infty)$ (b) $[-3, -2] \cup [2, 3]$
(c) $(-\infty, -2) \cup (2, 3)$ (d) $R - \{-3, -2, 2, 3\}$

43. The range of the function

$f(x)=\cos^2\dfrac{x}{4}+\sin\dfrac{x}{4}, x\in R$ is

(a) $\left[0, \dfrac{5}{4}\right]$ (b) $\left[1, \dfrac{5}{4}\right]$

(c) $\left(-1, \dfrac{5}{4}\right)$ (d) $\left[-1, \dfrac{5}{4}\right]$

44. Range of the function $f(x)=x^2+\dfrac{1}{x^2+1}$, is

(a) $[1, \infty]$ (b) $[2, \infty)$

(c) $\left[\dfrac{3}{2}, \infty\right)$ (d) $(-\infty, \infty)$

45. The inverse of $f(x) = (5 - (x - 8)^5)^{1/3}$ is

(a) $5 - (x - 8)^5$ (b) $8 + (5 - x^3)^{1/5}$
(c) $8 - (5 - x^3)^{1/5}$ (d) $(5 - (x - 8)^{1/5})^3$

46. Minimum value of function $f(x) = x^3(x^3+1)(x^3+2)(x^3+3) : x \in R$, is
(a) -2 (b) -1 (c) 1 (d) none

47. The domain of the function
$f(x) = \log_{10}\{1 - \log_{10}(x^2 - 5x + 10)\}$ is
(a) $(0, \infty)$ (b) $(0, 5)$
(c) $(-\infty, 0)$ (d) None of these

48. The range of the function
$f(x) = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$ where $[.]$ = G. I. F
(a) $\{\pi\}$ (b) $\left\{\frac{\pi}{2}\right\}$ (c) $\{2\pi\}$ (d) $\{0\}$

49. The domain of definition of the function, $f(x)$ given by the equation $2^x + 2^y = 2$ is
(a) $0 < x \le 1$ (b) $0 \le x \le 1$
(c) $-\infty < x \le 0$ (d) $-\infty < x < 1$

50. If $f : R \to R$ is a function satisfying the property $f(x+1) + f(x+3) = 2$ for all $x \in R$ then f is
(a) periodic with period 3
(b) periodic with period 4
(c) non periodic
(d) periodic with period 5

51. Let $f : R \to R - \{3\}$ be a function such that for some $p > 0$, $f(x+p) = \frac{f(x) - 5}{f(x) - 3}$ for all $x \in R$. Then, period of f is
(a) $2p$ (b) $3p$ (c) $4p$ (d) $5p$

52. The period of the function $f(x) = (-1)^{[x]}$ where $[.]$ = G.I.F
(a) 2 (b) 1 (c) 3 (d) 4

53. If $f(x) = x - \frac{1}{x}$, then number of solutions of $f(f(f(x))) = 1$ is
(a) 1 (b) 4 (c) 6 (d) 2

54. If $f(x) = x(x-1)$ is a function from $\left[\frac{1}{2}, \infty\right)$ to $\left[-\frac{1}{4}, \infty\right)$, then $\{x \in R : f^{-1}(x) = f(x)\}$ is
(a) null set
(b) $\{1\}$
(c) $\{0, 2\}$
(d) a set containing 3 elements

55. Let $f : \left[\frac{-\pi}{3}, \frac{2\pi}{3}\right] \to [0, 4]$ be a function defined as $f(x) = \sqrt{3}\sin x - \cos x + 2$ then $f^{-1}(x)$ is given by
(a) $\sin^{-1}\left(\frac{x-2}{2}\right) - \frac{\pi}{6}$ (b) $\sin^{-1}\left(\frac{x+2}{2}\right) + \frac{\pi}{6}$
(c) $\frac{2\pi}{3} - \cos^{-1}\left(\frac{x-2}{2}\right)$ (d) Does not exist

56. The value of the parameter α, for which the function $f(x) = 1 + \alpha x$, $\alpha \ne 0$, is the inverse of itself, is
(a) -2 (b) -1 (c) 1 (d) 2

57. If for nonzero x, $2f(x^2) + 3f\left(\frac{1}{x^2}\right) = x^2 - 1$ then $f(x^2) =$
(a) $\frac{3 + 2x^4 - x^2}{5x^2}$ (b) $\frac{3 - 2x^4 + x^2}{5x^2}$
(c) $\frac{3 - 2x^4 - x^2}{5x^2}$ (d) $\frac{3 + 2x^4 + x^2}{5x^2}$

58. If $g(x)$ is a polynomial satisfying $g(x)\,g(y) = g(x) + g(y) + g(xy) - 2$ for all real x and y and $g(2) = 5$, then $g(3)$ is equal to
(a) 10 (b) 24 (c) 21 (d) 15

59. Let $f\left(\frac{x+y}{2}\right) = \frac{1}{2}(f(x) + f(y))$ for real x and y. If $f'(x)$ exists and equals to -1 and $f(0) = 1$, then the value of $f(2)$ is
(a) 1 (b) -1 (c) $\frac{1}{2}$ (d) 2

60. A function $f : R \to R$ satisfies the equation $f(x)\,f(y) - f(xy) = x + y \;\; \forall\, x, y \in R$ and $f(1) > 0$, then
(a) $f(x)f^{-1}(x) = x^2 - 4$ (b) $f(x)f^{-1}(x) = x^2 - 6$
(c) $f(x)f^{-1}(x) = x^2 - 1$ (d) none of these

SOLUTIONS

1. (d) : $\sec x = 3 \Rightarrow \cos x = \frac{1}{3}$
which gives two values of x in each of $[0, 2\pi]$, $(2\pi, 4\pi]$, $(4\pi, 6\pi]$ and one value in $6\pi + \frac{3\pi}{2} = 15\frac{\pi}{2}$
$\therefore$ Greatest value of $n = 15$

2. (b) : $\sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}, \frac{7\pi}{2}$
$\therefore$ $K = 5$

3. (a) : $16(\sin^5 x + \cos^5 x) - 11(\sin x + \cos x) = 0$
$\Rightarrow$ $(\sin + \cos x)\{16(\sin^4 x - \sin^3 x\cos x + \sin^2 x\cos^2 x - \sin x\cos^3 x + \cos^4 x) - 11\} = 0$

$\Rightarrow$ $(\sin x + \cos x)\{16(1 - \sin^2 x \cos^2 x - \sin x \cos x) - 11\} = 0$

$\Rightarrow$ $(\sin x + \cos x)(4\sin x \cos x - 1)(4\sin x \cos x + 5) = 0$

As $4\sin x \cos x + 5 \neq 0$, we have

$\sin x + \cos x = 0$ or $4\sin x \cos x - 1 = 0$

The required values are $\pi/12$, $5\pi/12$, $9\pi/12$, $13\pi/12$, $17\pi/12$, $21\pi/12$, – they are 6 solutions on $[0, 2\pi]$.

4. (c) : $\sin^2\left(\frac{x}{2}\right) = \frac{x}{(2n+1)\pi}$ the graph of $\sin^2\left(\frac{x}{2}\right)$ will be above the x-axis and will be meeting the x-axis at $0, 2\pi, 4\pi, \ldots$ etc. It will attain maximum values at odd multiples of π *i.e.*, $\pi, 3\pi, \ldots (2n+1)\pi$. The last point after which graph of $y = \frac{x}{(2n+1)\pi}$ will stop cutting will be $(2n + 1)\pi$.

Total intersection $= 2(n + 1)$

5. (c) : Given equation is $\frac{1+\sin x}{\cos x} = 2\cos x$

$\Rightarrow$ $1 + \sin x = 2\cos^2 x = 2(1 - \sin^2 x)$

$\Rightarrow$ $2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow$ $(1 + \sin x)(2\sin x - 1) = 0$

$\Rightarrow$ $\sin x = -1$ or $1/2$

Now, $\sin x = -1 \Rightarrow \tan x$ and $\sec x$ not defined.

$\sin x = 1/2 \Rightarrow x = \pi/6$ or $5\pi/6$.

$\therefore$ The required number of solution is 2.

6. (a) : $|a + b + c| < |a| + |b| + |c|$

If a, b, c do not have same sign.

$\Rightarrow x^2 \sin x < 0 \quad \therefore \quad x \in (-\pi, 0)$

7. (d) : Given, $\log_{0.5} \sin x = 1 - \log_{0.5} \cos x$, $x \in [-2\pi, 2\pi]$

$\sin x > 0$ and $\cos x > 0$

$$\sin x \cos x = \frac{1}{2}$$

$\sin 2x = 1$, $2x \in [-4\pi, 4\pi]$

$\Rightarrow$ 4 solutions

8. (c) : $\because$ $\sin^2 x - 2\sin x + 5 = (\sin x - 1)^2 + 4 \geq 4$

$$\therefore\ 2^{\sqrt{\sin^2 x - 2\sin x + 5}} \geq 2^2 = 4$$

and $\sin^2 y \leq 1 \Rightarrow \frac{1}{4^{\sin^2 y}} \geq \frac{1}{4}$

$\therefore$ L.H.S. ≥ 1 and according to question L.H.S. ≤ 1 therefore, L.H.S. $= 1$

for which $\sin^2 x - 2\sin x + 5 = 4$

$\Rightarrow$ $(\sin x - 1)^2 = 0$

$\Rightarrow$ $\sin x = 1 \Rightarrow x = (2n + 1)\frac{\neq}{2}$

and $\sin^2 y = 1$ or $\cos y = 0$

$\Rightarrow$ $y = (2m + 1)\frac{\neq}{2}$

9. (c) : $\cos^2 10° - \cos 10° \cos 50° + \cos^2 50°$

$$= \frac{1}{2}\left[1 + \cos 20° - (\cos 60° + \cos 40°) + (1 + \cos 100°)\right]$$

$$= \frac{1}{2}\left[1 + \cos 20° - \frac{1}{2} - \cos 40° + 1 - \cos 80°\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} + \cos 20° - (2\cos 60° \cos 20°)\right] = \frac{3}{4}$$

10. (c) : Given equation is $\sec^2(a + 2)x + a^2 - 1 = 0$

$\Rightarrow$ $\tan^2(a + 2)x + a^2 = 0$

$\Rightarrow$ $\tan^2(a + 2)x = 0$ and $a = 0$

$$\Rightarrow \tan^2 2x = 0 \Rightarrow x = 0, \frac{\pi}{2}, \frac{-\pi}{2}$$

$\therefore$ $(0, 0)$, $(0, \pi/2)$, $(0, -\pi/2)$ are ordered pairs satisfying the equation.

11. (a) : $a\tan^2 x + b = c(1 + \tan^2 x)$

$$\Rightarrow \tan^2 x = \left(\frac{c-b}{a-c}\right), \tan^2 y = \left(\frac{d-a}{b-d}\right)$$

$$\therefore\ \frac{a^2}{b^2} = \frac{\tan^2 y}{\tan^2 x} = \frac{(a-d)(c-a)}{(b-c)(d-b)}$$

12. (a) : $\tan\theta/2 = \sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)} = \sqrt{\frac{1-\left(\frac{a\cos\phi + b}{a + b\cos\phi}\right)}{1+\left(\frac{a\cos\phi + b}{a + b\cos\phi}\right)}}$

$$= \sqrt{\frac{(a-b)(1-\cos\phi)}{(a+b)(1+\cos\phi)}} = \sqrt{\frac{(a-b)}{(a+b)}}\tan(\phi/2)$$

13. (b) : $\cos\alpha + \cos(\alpha + \beta) + \ldots + \cos(\alpha + (n - 1)\beta)$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cos\left(\alpha + \frac{(n-1)\beta}{2}\right)$$

14. (a) : $\frac{s-a}{11} = \frac{s-b}{12} = \frac{s-c}{13} = \frac{s}{36}$

On solving we get, $\tan^2(A/2) = \frac{13}{33} \Rightarrow \lambda = 1155$

15. (a) : Given, $2\sin x + 2\sin^2 x - 1 = 1$

or $\sin^2 x + \sin x - 1 = 0$

$$\therefore\ \sin x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1+\sqrt{5}}{2}$$

$$\Rightarrow \sin^2 x = \frac{3-\sqrt{5}}{2} \Rightarrow \cos^2 x = \frac{\sqrt{5}-1}{2}$$

$$\therefore \cos^2 x(1+\cos^2 x) = \frac{\sqrt{5}-1}{2} \times \frac{\sqrt{5}+1}{2} = 1$$

16. (d) : $1 + \cos px + 1 + \cos qx = 2$

$$\Rightarrow \cos\left(\frac{p+q}{2}\right)x \cos\left(\frac{p-q}{2}\right)x = 0$$

$$\Rightarrow x = \frac{(2n+1)\pi}{p+q} \text{ or } \frac{(2n+1)\pi}{p-q}$$

for $n = 0, \pm 1, \pm 2, \ldots\ldots$

forms an A.P. with common difference $\frac{2\pi}{p+q}$ or $\frac{2\pi}{p-q}$

17. (a) : The given condition can be written as $(\cos^2\alpha + \sin^2\alpha)^3 - 3\sin^2\alpha \cos^2\alpha(\cos^2\alpha + \sin^2\alpha) + k\sin^2 2\alpha = 1$

$$\Rightarrow \left(-\frac{3}{4}\right)\sin^2 2\alpha + k\sin^2 2\alpha = 0$$

Showing that $k = \frac{3}{4}$.

18. (c) : We have, $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$

The value of θ lying between $\frac{3\pi}{2}$ and 2π and satisfying these two is $\frac{7\pi}{4}$. Therefore the most general solution is $\theta = 2n\pi + 7\pi/4$ where $n \in Z$

19. (a) : $\because 1 + |\cos x| + \cos^2 x \ldots. = \frac{1}{1-|\cos x|}$

$$\Rightarrow \frac{1}{8^{1-|\cos x|}} = 4^3 \Rightarrow 2^{\frac{3}{1-|\cos x|}} = 2^6$$

$$\Rightarrow \frac{3}{1-|\cos x|} = 6 \Rightarrow 1-|\cos x| = \frac{1}{2}$$

$$|\cos x| = \frac{1}{2} \Rightarrow \cos x = \pm\frac{1}{2}$$

For least positive value of x, $x = \frac{\pi}{3}$

20. (c) : $A(z) = A\left(\frac{x+y}{1+xy}\right)$

$$= \left[\frac{1+xy}{(1-x)(1-y)}\right]\begin{bmatrix} 1 & -\left(\frac{x+y}{1+xy}\right) \\ -\left(\frac{x+y}{1+xy}\right) & 1 \end{bmatrix}$$

$\therefore A(x).A(y) = A(z)$

21. (c)

22. (c) : $(A^{-1}BA)^2 = (A^{-1}BA)(A^{-1}BA) = A^{-1}B(AA^{-1})BA$
$= A^{-1}BIBA = A^{-1}B^2A$

$\Rightarrow (A^{-1}BA)^3 = (A^{-1}B^2A)^2 = A^{-1}B^2(AA^{-1})BA$
$= A^{-1}B^2IBA = A^{-1}B^3A$ and so on

$\Rightarrow (A^{-1}BA)^n = A^{-1}B^nA$

23. (b) : We have, $A = iB$

$$\Rightarrow A^2 = (iB)^2 = i^2B^2 = -B^2 = -\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = -2B$$

$\Rightarrow A^4 = (-2B)^2 = 4B^2 = 4(2B) = 8B$

$\Rightarrow (A^4)^2 = (8B)^2 \Rightarrow A^8 = 64B^2 = 128B$

24. (c) : $p + q + r = 0$

$\Rightarrow p^3 + q^3 + r^3 = 3pqr$

$$\text{Now, } \begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr(a^3 + b^3 + c^3 - 3abc)$$

$$= pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \Rightarrow k = pqr$$

25. (c) : Since $P^2 = I - P$ (given) ...(i)

$\Rightarrow P^3 = P(I - P)$

$\Rightarrow P - P^2 = P - (I - P)$ (using(i))

$\therefore P^3 = 2P - I$ ...(ii)

Similarly, $P^4 = 2P^2 - P = 2I - 3P$ and $P^5 = 5P - 3I$ and $P^6 = 5P^2 - 3P = 5I - 8P$

26. (b) : $100A + 80 + 8 = 72\lambda_1$

$600 + 10B + 8 = 72\lambda_2$

$800 + 60 + C = 72\lambda_3$, $\lambda_1, \lambda_2, \lambda_3 \in I$

$$\text{Let } \Delta = \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$$

$$= \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 72\lambda_1 & 72\lambda_2 & 72\lambda_3 \end{vmatrix} (R_3 \leftrightarrow R_3 + 10R_2 + 100R_1) \quad \text{...(i)}$$

Now, $A88$ is divisible by 72

$\Rightarrow A88$ is divisible by 9

$\therefore A = 2$

Also, $6B8$ is divisible by 9

Substituting these values in (i) we get Δ is divisible by $144 \Rightarrow B = 4$

27. (b) : $A \cdot B = O \Rightarrow A \cdot B \cdot B^{-1} = O \cdot B^{-1}$

$\Rightarrow \quad A \cdot I = O \Rightarrow A = O$

28. (a) : $x^k y^k z^k \begin{vmatrix} 1 & ar & a^2r^2 \\ 1 & ar^2 & a^2r^4 \\ 1 & ar^3 & a^3r^6 \end{vmatrix}$

$a^{3(k+1)} . r^{3(2k+1)}[r-1](r^4-1)-(r^2-1)^2]$

$\Rightarrow \quad k = -1$

29. (c) : $a_{ji} = j^2 - i^2 = -(i^2 - j^2) = -a_{ij}$

30. (b) : $A^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, A^4 = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$

$A^8 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}, A^{16} = \begin{bmatrix} 256 & 0 \\ 0 & 256 \end{bmatrix}$

31. (d) : $AB = \begin{bmatrix} a & 2b \\ 3a & 4b \end{bmatrix}, BA = \begin{bmatrix} a & 2a \\ 3b & 4b \end{bmatrix}$

$AB = BA \Rightarrow a = b$

32. (a) : $(B^TAB)^T = B^TA^T(B^T)^T = B^TA^TB$

$= B^TAB$ iff A is symmetric

$\therefore \quad B^TAB$ is symmetric iff A is symmetric

Also, $(B^TAB)^T = B^TA^TB = (-B)A^TB$

$\therefore \quad B^TAB$ is not skew symmetric if B is skew symmetric

33. (d) : $2ABC$ is not defined

$\therefore$ there is no solution

34. (d) : Since $AB = B$ and $BA = A$

$\therefore \quad A$ and B both are idempotent

$(A - B)^2 = A^2 - AB - BA + B^2$

$\qquad = A - B - A + B = 0$

$\therefore \quad A - B$ is nilpotent

35. (d) : $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}$

$BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}$

If $AB = BA$, then $a + 2c = a + 3b$

$\Rightarrow \quad 2c = 3b \Rightarrow b \neq 0$

Now, $b + 2d = 2a + 4b$

$\Rightarrow \quad 2a - 2d = -3b$

$$\therefore \quad \frac{a-d}{3b-c} = \frac{-\frac{3}{2}b}{3b-\frac{3}{2}b} = -1$$

36. (b) : $\left(f(\alpha)\right)^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(-\alpha)$

37. (d) : $|A^{1003} - 5A^{1002}| = |A^{1002}(A - 5I)|$

$= |A^{1002}|\,|A - 5I| = |A|^{1002}\,|A - 5I|$

$= 1 \times \begin{vmatrix} 0 & -6 \\ 1 & -6 \end{vmatrix} = 6$

38. (c): Given that $f(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

39. (d) : $f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right) \in R$

$\Leftrightarrow -1 \le \log_2\left(\frac{x^2}{2}\right) \le 1 \Leftrightarrow \frac{1}{2} \le \frac{x^2}{2} \le 2$

$\Leftrightarrow 1 \le x^2 \le 4 \;\Leftrightarrow x \in [-2, -1] \cup [1, 2]$

40. (a) : $-1 \le \sin 3x \le 1$

41. (a) : Let $y = \frac{x^2 - x + 1}{x^2 + x + 1}$

$\Rightarrow \quad yx^2 + yx + y = x^2 - x + 1$

$\Rightarrow \quad (y - 1)x^2 + (y + 1)x + (y - 1) = 0$

Now, $x \in R \Rightarrow$ Discriminant ≥ 0

$\Rightarrow \quad (y + 1)^2 - 4(y - 1)^2 \ge 0$

$\Rightarrow \quad -3y^2 + 10y - 3 \ge 0$

$\Rightarrow 3y^2 - 10y + 3 \le 0 \Rightarrow (3y - 1)(y - 3) \le 0 \Rightarrow \frac{1}{3} \le y \le 3$

$\therefore \quad \text{Range} = \left[\frac{1}{3}, 3\right]$

42. (b) : $5|x| - x^2 - 6 \ge 0 \;\Rightarrow x^2 - 5|x| + 6 \le 0$

when $x < 0$, $x^2 + 5x + 6 \le 0$, $-3 \le x \le -2$

when $x > 0$, $x^2 - 5x + 6 \le 0$, $2 \le x \le 3$

$x = 0$ will not satisfy the condition.

Domain is $[-3, -2] \cup [2, 3]$.

43. (d) : $f(x) = 1 - \sin^2\frac{x}{4} + \sin\frac{x}{4}$

$$= -\left\{\sin^2\frac{x}{4} - \sin\frac{x}{4}\right\} + 1 = -\left\{\left(\sin\frac{x}{4} - \frac{1}{2}\right)^2 - \frac{1}{4}\right\} + 1$$

Maximum $f(x) = \frac{5}{4}$

Minimum $f(x) = \frac{5}{4} - \left(-1 - \frac{1}{2}\right)^2 = \frac{5}{4} - \frac{9}{4} = -1$

Range of $f(x) = \left[-1, \frac{5}{4}\right]$

44. (a) : Here, $f(x) = x^2 + 1 + \frac{1}{x^2+1} - 1$

$\because \quad x^2 + 1 + \frac{1}{x^2+1} \geq 2 \Rightarrow x^2 + \frac{1}{x^2+1} \geq 1$

$\therefore \quad f(x) \in [1, \infty)$

45. (b) : Let $y = f(x) = (5 - (x - 8)^5)^{1/3}$

Then $y^3 = 5 - (x - 8)^5 \Rightarrow (x - 8)^5 = 5 - y^3$

$\Rightarrow x = 8 + (5 - y^3)^{1/5}$

Let, $z = g(x) = 8 + 5(5 - x^3)^{1/5}$

Now, $f(g(x)) = (5 - [5 - x^3)^{1/5}]^5)^{1/3} = (5 - 5 + x^3)^{1/3} = x$

Similarly, we can show that $g(f(x)) = x$.

Hence, $g(x) = 8 + (5 - x^3)^{1/5}$ is the inverse of $f(x)$.

46. (b) : Let $t = x^3(x^3+3); t = (x^3 + \frac{3}{2})^2 - \frac{9}{4} \in [-\frac{9}{4}, \infty)$

$f(x) = g(t) = t(t + 2) = (t + 1)^2 - 1$ is least when $t = -1$ and $-1 \in [-\frac{9}{4}, \infty) \therefore$ min $f(x) = -1$

47. (b) : For function $f(x)$ to be defined we have $x^2 - 5x + 10 > 0$ ...(i) and $1 - \log_{10}(x^2 - 5x + 10) > 0$ ...(ii)

Now, (ii) $\Rightarrow \log_{10}(x^2 - 5x + 10) < 1 \Rightarrow x^2 - 5x + 10 < 10$

$\Rightarrow x^2 - 5x < 0 \Rightarrow x(x - 5) < 0 \Rightarrow 0 < x < 5$ ...(A)

Again, $x^2 - 5x + 10 > 0$ for all x, ...(B)

Since the discriminant of the corresponding equation $x^2 - 5x + 10 = 0$ is negative, so that the roots of the equation are imaginary.

Combining (A) and (B), we find that the domain of $f(x)$ is (0, 5).

48. (a) : Let $y_1 = \sin^{-1}\left[x^2 + \frac{1}{2}\right]$ and $y_2 = \cos^{-1}\left[x^2 - \frac{1}{2}\right]$

Then, $y = y_1 + y_2$.

Now, $y_1 = \sin^{-1}\left[x^2 + \frac{1}{2}\right]$ is defined

if $-1 \leq \left[x^2 + \frac{1}{2}\right] \leq 1 \Rightarrow -1 \leq x^2 + \frac{1}{2} < 2 \Rightarrow -\frac{3}{2} \leq x^2 < \frac{3}{2}$ ...(i)

Again, $y_2 = \cos^{-1}\left[x^2 - \frac{1}{2}\right]$ is defined

If $-1 \leq \left[x^2 - \frac{1}{2}\right] \leq 1 \Rightarrow -1 \leq x^2 - \frac{1}{2} < 2 \Rightarrow -\frac{1}{2} \leq x^2 < \frac{5}{2}$ ...(ii)

Taking the intersection of (i) and (ii), we find that

$-\frac{1}{2} \leq x^2 < \frac{3}{2} \Rightarrow 0 \leq x^2 < \frac{3}{2}$, since x^2 cannot be negative.

Now, for x^2 so that $\frac{1}{2} \leq x^2 + \frac{1}{2} \leq 1$ and $-\frac{1}{2} \leq x^2 - \frac{1}{2} \leq 0$

We have $Y = \sin^{-1}(0) + \cos^{-1}(-1) = 0 + \pi - \cos^{-1}(1) = \pi$

Similarly for $\frac{1}{2} \leq x^2 < \frac{3}{2}$, we have

$y = \sin^{-1}(1) + \cos^{-1}(0) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

Hence, the range of the given function is $[\pi]$.

49. (d) : It is given that $2^x + 2^y = 2 \ \forall \ x, y \in R$

Therefore, $2^x = 2 - 2^y < 2 \Rightarrow 0 < 2^x < 2$

Taking log for both side with base 2.

$\Rightarrow \log_2 0 < \log_2 2^x < \log_2 2$

Hence domain is $-\infty < x < 1$

50. (b) : $f(x + 1) = f(x + 5)$

51. (c) : 3 does not belong to the range of f implies 2 also cannot belong to range of f because, if $f(x) = 2$ for some $x \in R$. Then $f(x+p) = \frac{2-5}{2-3} = 3$ which is not in the range of f. Hence 2 and 3 are not in the range of f. If $f(x + 2p) = f(x)$, this implies $f(x) = f(x + p + p)$

$$= \frac{f(x+p)-5}{f(x+p)-3} = \frac{\frac{f(x)-5}{f(x)-3} - 5}{\frac{f(x)-5}{f(x)-3} - 3}$$

$$= \frac{-4f(x)+10}{-2f(x)+4} = \frac{2f(x)-5}{f(x)-2}$$

so that $[f(x) - 2]^2 = -1$ which is absurd. Therefore, $2p$ is not a period.

Again, $f(x+3p) = \frac{2f(x+p)-5}{f(x+p)-2} = \frac{3f(x)-5}{f(x)-1} \neq f(x)$

Now, $f(x + 4p) = f(x + 3p + p)$

$$=\frac{f(x+3p)-5}{f(x+3p)-3}=\frac{\frac{3f(x)-5}{f(x)-1}-5}{\frac{3f(x)-5}{f(x)-1}-3}=\frac{-2f(x)}{-2}=f(x)$$

Therefore, $4p$ is a period.

52. (a) : Given: $f(x) = (-1)^{[x]}$.

First of all, we sketch the graph of $f(x)$ with the help of piecewise defined functions as follows:

$$f(x)=(-1)^{[x]}=\begin{cases}1; & -2\le x<-1\\ -1; & -1\le x<0\\ 1; & 0\le x<1\\ -1: & 1\le x<2\\ 1; & 2\le x<3.\end{cases}$$

$\because$ The function $f(x)$ repeats its value after the least interval of 2.

Therefore, the function $f(x)$ is periodic with period 2.

53. (b) : $f(x)=x-\frac{1}{x}, \Rightarrow f(f(x))=\frac{x^4-3x^2+1}{x(x^2-1)}$

Since, we have $f(f(f(x)))=1\Rightarrow f(f(x))=f^{-1}(1)=\frac{1+\sqrt{5}}{2}$

$\Rightarrow$ 2 values exist

or $f^{-1}(1)=\frac{1-\sqrt{5}}{2}$ $\Rightarrow$ 2 values exist

54. (c) : $\{x \in R : f^{-1}(x) = f(x)\} = \{x \in R : f f(x) = x\}$

$f(f(x)) = f(x)[f(x) - 1] = [x(x - 1)][x(x - 1) - 1]$
$= x(x - 1)[x^2 - x - 1]$

$f(f(x)) = x \Rightarrow x(x - 1)(x^2 - x - 1) = x$

$\Rightarrow x(x^3 - 2x^2) = 0 \Rightarrow x = 0, 2$

55. (c) : We have, $f(x)=2\sin\left(x-\frac{\pi}{6}\right)+2$

$\because$ f is one–one and onto $\therefore$ f is invertible

Now, $fof^{-1}(x) = x \Rightarrow 2\sin\left(f^{-1}(x)-\frac{\pi}{6}\right)+2=x$

$$f^{-1}(x)=\sin^{-1}\left(\frac{x}{2}-1\right)+\frac{\pi}{6}\quad\left(\because\left|\frac{x}{2}-1\right|\le 1\,\forall x\in[0,4]\right)$$

Also, $\sin^{-1}\alpha + \cos^{-1}\alpha = \frac{\neq}{2}$

$$\Rightarrow f^{-1}(x)=\frac{\pi}{2}-\cos^{-1}\frac{x-2}{2}+\frac{\pi}{6}=\frac{2\pi}{3}-\cos^{-1}\left(\frac{x-2}{2}\right)$$

56. (b) : Let $y=f(x)=1+\alpha x \Rightarrow x=\frac{y-1}{\alpha}$

$\Rightarrow f^{-1}(x)=\frac{x-1}{\alpha}$

Now, $f(x)=f^{-1}(x)\Rightarrow \frac{x-1}{\alpha}=1+\alpha x \Rightarrow x-1=\alpha+\alpha^2 x$

Equating the coefficient of x, we get

$\alpha^2 = 1$ and $\alpha = -1 \Rightarrow \alpha = -1$

57. (c) : We have, $2f(x^2)+3f\left(\frac{1}{x^2}\right)=x^2-1$

$$\Rightarrow 4f(x^2)+6f\left(\frac{1}{x^2}\right)=2x^2-2 \quad \text{...(i)}$$

Also, $2f\left(\frac{1}{x^2}\right)+3f(x^2)=\frac{1}{x^2}-1$

$$\Rightarrow 9f(x^2)+6f\left(\frac{1}{x^2}\right)=\frac{3}{x^2}-3 \quad \text{...(ii)}$$

$$\text{(ii)}-\text{(i)}\Rightarrow 5f(x^2)=\frac{3}{x^2}-2x^2-1\Rightarrow f(x^2)=\frac{3-2x^4-x^2}{5x^2}$$

58. (a) : We have, $g(x)\, g(y) = g(x)+ g(y)+g(xy)-2$ ...(i)

Putting $x = 1$, $y = 2$ in (i), we have

$g(1)\, g(2) = g(1) + g(2) + g(2) - 2$

$\Rightarrow 5g(1) = 8 + g(1) \therefore\ g(1) = 2$

Also, replacing y by $\frac{1}{x}$ in (i), we get

$$g(x)g\left(\frac{1}{x}\right)=g(x)+g\left(\frac{1}{x}\right)+g(1)-2$$

$$\Rightarrow g(x)g\left(\frac{1}{x}\right)=g(x)+g\left(\frac{1}{x}\right)$$

$\Rightarrow g(x) = 1 \pm x^n$ ...(ii)

Put $x = 2$ in (ii), we get $\pm 2^n = 2^2$

Taking +ve sign, we set $n = 2$

$\therefore\ g(x) = 1 + x^2 \Rightarrow g(3) = 1 + 3^2 = 10$

59. (b)

60. (c) : Taking $x = y = 1$, we get

$f(1)f(1) - f(1) = 2 \Rightarrow f^2(1) - f(1) - 2 = 0$

$\Rightarrow (f(1) - 2)(f(1) + 1) = 0 \Rightarrow f(1) = 2$ (as $f(1) > 0$)

Taking $y = 1$, we get

$f(x).f(1) - f(x) = x + 1$

$\Rightarrow f(x) = x + 1 \Rightarrow f^{-1}(x) = x - 1$

$\therefore\ f(x).f^{-1}(x) = x^2 - 1$

JEE ADVANCED

PRACTICE PAPER 2020

PAPER - I

SECTION-1

ONE OR MORE THAN ONE OPTION CORRECT TYPE

1. Which of the following set of values of 'x' satisfies the equation $2^{(2\sin^2 x-3\sin x+1)}+2^{(2-2\sin^2 x+3\sin x)}=9$

(a) $x=n\pi\pm\frac{\pi}{6}, n\in I$ (b) $x=n\pi\pm\frac{\pi}{3}, n\in I$

(c) $x=n\pi,\ n\in I$ (d) $x=2n\pi+\frac{\pi}{2}, n\in I$

2. If α, β ($\alpha<\beta$) are 2 roots of $(6x+1)x=1+\left[\cos\frac{\pi}{4}\right]$ then $\int_0^{2\alpha}\sin\left(\frac{\pi[x]}{2}\right)dx+\int_0^{3\beta}\cos(\pi[x])dx=$ __ , where [.] denotes greatest integer function.

(a) $\alpha+\beta$ (b) 2

(c) 0 (d) $[2\alpha+9\beta]$

3. If three numbers are chosen randomly from the set $\{1, 3, 3^2, ..., 3^n\}$ without replacement, then the probability that they form an increasing geometric progression is

(a) $\frac{3}{2n}$ if n is odd

(b) $\frac{3}{2n}$ if n is even

(c) $\frac{3n}{2(n^2-1)}$ if n is even

(d) $\frac{3n}{2(n^2-1)}$ if n is odd

4. The general solution of the differential equation $(1+y^2)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0$ is $2xe^{f(y)}=e^{2f(y)}+c$, then the area of the region bounded by the curves $x=f(y)$, $y=\pm\sqrt{3}$ and y-axis is

(a) $\frac{\pi}{\sqrt{3}}-\log 2$ (b) $\frac{2\pi}{\sqrt{3}}-\log 4$

(c) $\frac{\pi}{\sqrt{3}}+\log 2$ (d) $\frac{2\neq}{\sqrt{3}}$

5. Given a real valued function f such that

$$f(x)=\begin{cases}\dfrac{\tan^2 x}{(x^2-[x]^2)}, & \text{for } x>0\\ 1, & \text{for } x=0\\ \sqrt{\{x\}\cot\{x\}}, & \text{for } x<0\end{cases}$$

Where $[x]$ is the integral part and $\{x\}$ is the fractional part of x, then

(a) $\lim_{x\to0}f(x)=1$ (b) $\lim_{x\to0^-}f(x)=\sqrt{\cot 1}$

(c) $\cot^{-1}\left(\lim_{x\to0^-}f(x)\right)^2=1$

(d) f is continuous at $x=0$

6. A vector of magnitude 2 along a bisector of the angle between the two vectors $2\hat{i}-2\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}-2\hat{k}$ is

(a) $\frac{2}{\sqrt{10}}\left(3\hat{i}-\hat{k}\right)$ (b) $\frac{1}{\sqrt{26}}\left(\hat{i}-4\hat{j}+3\hat{k}\right)$

(c) $\frac{2}{\sqrt{26}}\left(\hat{i}-4\hat{j}+3\hat{k}\right)$ (d) $\frac{2}{\sqrt{5}}\left(\hat{i}-3\hat{j}\right)$

7. Let $f(x)=\frac{x}{1+x^2}$ and $g(x)=\frac{e^{-x}}{1+[x]}$, where [·] is the greatest integer less than or equal to x, then

(a) Domain $(f+g)=R-[-2, 0)$

(b) Domain $(f-g)=R-[-1, 0)$

(c) Range $f \cap$ Range $g = \left[-2, \frac{1}{2}\right]$

(d) Range $g = R - \{0\}$

SECTION-2

INTEGER ANSWER TYPE

8. The number of acute triangles whose vertices are chosen from the vertices of a rectangular block are

9. Tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is meet by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$, then the x-coordinate of the point of intersection of these tangents is

10. Let $a_1, a_2, \ldots\ldots, a_n$ be an A.P. with common difference $\pi/6$ and assume

$$\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \ldots + \sec a_{n-1} \sec a_n = k(\tan a_n - \tan a_1).$$

Find the value of k.

11. If $f(x) = x^2 - x + 1, x \geq \frac{1}{2}$ and $g(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$ are two functions, then the number of solutions of the equation $x^2 - x + 1 = \frac{1}{2} + \sqrt{\left(x - \frac{3}{4}\right)}$ is

12. If $f(x + y) = f(x) f(y)$ and $f(x) = 1 + xg(x)H(x)$ where $\lim_{x \to 0} g(x) = 2$, $\lim_{x \to 0} H(x) = 3$, then $f'(x) = Kf(x)$ if $K =$

SECTION-3

MATRIX MATCH TYPE

13. Match the following.

	Column-I		Column-II
(A)	Radius of the circle passing through (3, 4) and (5, 2) having centre on $y = 2x$ is	(p)	4
(B)	Length of the common chord of $2y^2 = 3(x + 1)$, $x^2 + y^2 + 2x = 0$ is	(q)	$\sqrt{52}$
(C)	Product of length of perpendiculars drawn from the two foci to the tangent at any point on the ellipse $25x^2 + 4y^2 = 100$ is	(r)	$\sqrt{3}$

14. Match the following.

	Column-I		Column-II
(A)	If $I_1 = \int_0^1 x^4(1-x)^5\,dx$, $I_2 = \int_0^1 x^5(1-x)^5\,dx$ then $\frac{I_2}{I_1} =$	(p)	$\frac{1}{2}$
(B)	If $I_1 = \int_0^{\pi} \cos^4 x \sin^6 x\,dx$, $I_2 = \int_0^{2\pi} \cos^4 x \sin^4 x\,dx$ then $\frac{I_1}{I_2} =$	(q)	$\frac{1}{3}$
(C)	If $f(x) = \frac{\sin^2 x}{(1 - e^{-3x})\sin^2 2x}$ then $\int_{-\pi/4}^{\pi/4} f(x)dx =$	(r)	$\frac{1}{4}$
(D)	$\int_0^{\pi/2} (\sqrt{\sin x} + \sqrt{\cos x})^{-4}\,dx =$	(s)	$\frac{5}{11}$

15. Match the following:

	Column-I		Column-II
(A)	The no. of solutions of the equation $\lvert\cos x\rvert = 2[x]$, (where $[\cdot]$ is greatest integer function) is	(p)	8
(B)	The no. of solutions of the equation $2^{\cos x} = \lvert\sin x\rvert$ in $[-2\pi, 2\pi]$ is	(q)	4
(C)	The no. of solutions of the equation $\sin^3 x \cos x + \sin^2 x \cos^2 x + \sin x \cos^3 x = 1$ in $[0, 2\pi]$ are	(r)	0
(D)	The number of possible values of k if fundamental period of $\sin^{-1}(\sin kx)$ is $\pi/2$ is	(s)	2

PAPER-II

SECTION-1

SINGLE OPTION CORRECT TYPE

1. Through the point $P(h, k, l)$ a plane is drawn at right angles to OP to meet co-ordinate axes at A, B and C. If $OP = p$, then the area of the ΔABC is

(a) $\frac{p^5}{2hkl}$ (b) $\frac{p^5}{hkl}$ (c) $\frac{p^3}{2hkl}$ (d) $\frac{p^3}{hkl}$

2. Sixteen players $s_1, s_2, \ldots, s_{16}$ play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pair. Assume that all the players are of equal strength. The probability that "exactly one of the two players s_1 and s_2 is among the eight winners" is

(a) 4/15 (b) 7/15 (c) 8/15 (d) 9/15

3. The solution of the differential equation

$$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$$ is

(a) $\sqrt{x^2 + y^2} = a\cos\left\{c + \tan^{-1}\frac{y}{x}\right\}$

(b) $\sqrt{x^2 + y^2} = a\sin\left\{c + \tan^{-1}\frac{y}{x}\right\}$

(c) $\sqrt{x^2 + y^2} = a\sin\left\{c + \tan^{-1}\frac{x}{y}\right\}$

(d) $\sqrt{x^2 + y^2} = a\cos\left\{c + \tan^{-1}\frac{x}{y}\right\}$

4. In a certain test there are n questions. In this test 2^{n-i} students gave wrong answers to atleast i questions, where $i = 1, 2, 3, \ldots, n$. If the total number of wrong answers given is 2047, then n is

(a) 10 (b) 11 (c) 12 (d) 13

5. Let three matrices $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$, then $tr(A) + tr\left(\frac{ABC}{2}\right) + tr\left(\frac{A(BC)^2}{4}\right) + tr\left(\frac{A(BC)^3}{8}\right) + \ldots\ldots + \infty =$

(a) 6 (b) 9
(c) 12 (d) none of these

6. If a, b, c and d are distinct positive real numbers such that a and b are the roots of $x^2 - 10cx - 11d = 0$ and c and d are the roots of $x^2 - 10ax - 11b = 0$, then the value of $a + b + c + d$ is

(a) 1110 (b) 1010 (c) 1101 (d) 1210

7. The square $ABCD$ where $A(0, 0)$, $B(2, 0)$, $C(2, 2)$, $D(0, 2)$ undergoes the following transformations successively

(i) $f_1(x, y) = (y, x)$ (ii) $f_2(x, y) = (x + 3y, y)$

(iii) $f_3(x, y) = \left(\frac{x-y}{2}, \frac{x+y}{2}\right)$. Then the final figure is

(a) a square (b) a rectangle
(c) a parallelogram (d) a rhombus

SECTION-2

ONE OR MORE THAN ONE OPTION CORRECT TYPE

8. If $\cos x + \cos y = a$, $\cos 2x + \cos 2y = b$, $\cos 3x + \cos 3y = c$, then

(a) $\cos^2 x + \cos^2 y = 1 + \frac{b}{2}$

(b) $\cos x \cos y = \frac{a^2}{2} - \left(\frac{b+2}{4}\right)$

(c) $2a^3 + c = 3a(1 + b)$

(d) $a + b + c = 3abc$

9. If $f(x) = \int_x^{x^2} \frac{dt}{(\log t)^2}$, $x \neq 0$, $x \neq 1$, then $f(x)$ is

(a) monotonically increasing in $(2, \infty)$
(b) monotonically increasing in $(1, 2)$
(c) monotonically decreasing in $(2, \infty)$
(d) monotonically decreasing in $(0, 1)$

10. Let $A = \int_{e^{-1}}^{\tan x} \frac{t\,dt}{t^2 + 1}$ and $B = \int_{e^{-1}}^{\cot x} \frac{dt}{t(t^2 + 1)}$, then

(a) At $x = \frac{\pi}{4}$, $A + B = 1$

(b) $A + B = 1$ for all x in $\left(0, \frac{\pi}{2}\right)$

(c) $A + B = 1$ for all x in $\left(0, \frac{\pi}{4}\right)$ and 2 for all x in $\left(\frac{\pi}{4}, \pi\right)$

(d) $A = B$ for all x

11. If $h(x)=3f\left(\frac{x^2}{3}\right)+f(3-x^2), \forall x \in (3,4)$, where $f''(x)>0,\ \forall\ x\in(-3,\ 4)$, then $h(x)$ is
(a) increasing in (3/2, 4)
(b) increasing in (–3/2, 0)
(c) decreasing in (0, 3/2)
(d) None of these

12. The eccentric angles of extremities of a chord of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are θ_1 and θ_2. If this chord passes through the focus, then
(a) $\tan\frac{\theta_1}{2}\cdot\tan\frac{\theta_2}{2}+\frac{1-e}{1+e}=0$
(b) $\cos\left(\frac{\theta_1-\theta_2}{2}\right)=e\cos\left(\frac{\theta_1+\theta_2}{2}\right)$
(c) $\cot\frac{\theta_1}{2}\cdot\cot\frac{\theta_2}{2}=\frac{e+1}{e-1}$
(d) None of these

13. The normal at a general point $(a,\ b)$ on curve makes an angle θ with x-axis which satisfies $b(-a^2\tan\theta-\cot\theta)=a(b^2+1)$. The equation of curve can be
(a) $y = e^{x^2/2} + c$ (b) $\log(ky^2) = x^2$
(c) $y = ke^{x^2/2}$ (d) $x^2 - y^2 = k$

14. If $S_r=\begin{vmatrix} 2r & x & n(n+1) \\ 6r^2-1 & y & n^2(2n+3) \\ 4r^3-2nr & z & n^3(n+1)\end{vmatrix}$, then the value $\sum_{r=1}^{n} S_r$ is independent of
(a) x (b) y (c) n (d) z

SECTION-3
COMPREHENSION TYPE

Paragraph for Question No. 15 and 16

Let A, B, C be three sets of complex numbers as defined below,
$A = \{z : \text{Im } z \geq 1\}$; $B = \{z : |z - 2 - i| = 3\}$
$C = \{z : \text{Re}((1 - i)z) = \sqrt{2}\}$

15. The number of elements in the set $A \cap B \cap C$ is
(a) 0 (b) 1 (c) 2 (d) ∞

16. Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between
(a) –6 and 3 (b) –3 and 6
(c) –6 and 6 (d) –3 and 9

Paragraph for Question No. 17 and 18

Define a function $\phi : N \to N$ as follows : $\phi(1) = 1$, $\phi(p^n) = p^{n-1}$, if p is prime and $n \in N$. $\phi(mn) = \phi(m)\,\phi(n)$ if m and n are relatively prime natural numbers.

17. $\phi(8n + 4)$ where $n \in N$ is equal to
(a) $\phi(4n + 2)$ (b) $\phi(2n + 1)$
(c) $2\phi(2n + 1)$ (d) $4\phi(2n + 1)$

18. The number of natural numbers 'n' such that $\phi(n)$ is odd is
(a) 1 (b) 2 (c) 3 (d) 4

SOLUTIONS
PAPER-I

1. **(a, d) :** $2^{(2\sin^2 x-3\sin x+1)}+2^{3-(2\sin^2 x-3\sin x+1)}=9$
Let $2^{(2\sin^2 x - 3\sin x + 1)} = t$
$\Rightarrow t+\frac{8}{t}=9 \Rightarrow t^2-9t+8=0 \Rightarrow t = 1,\ 8$
$\Rightarrow 2\sin^2 x - 3\sin x + 1 = 3$ or $2\sin^2 x - 3\sin x + 1 = 0$
$\Rightarrow \sin x=-\frac{1}{2}, \sin x=\frac{1}{2}, \sin x=1$

2. **(b,d) :** $(6x+1)x=1+\left[\cos\frac{\pi}{4}\right]$
$\Rightarrow\ 6x^2+x-1=0 \Rightarrow\ x=\frac{-1}{2}, \frac{1}{3}$; $\alpha=-\frac{1}{2},\ \beta=\frac{1}{3}$
$\therefore\ I=\int_0^{-1}\sin\frac{\pi[x]}{2}dx+\int_0^1\cos(\pi[x])dx=\int_{-1}^{0}1dx+\int_0^1 1dx=2$
Also $[2\alpha + 9\beta] = [-1 + 3] = 2$

3. **(a,c) :** Number of triplets
$(3^r, 3^{r+1}, 3^{r+2})$ $(0 \leq r \leq n - 2)$ is $n - 1$
Number of triplets
$(3^r, 3^{r+2}, 3^{r+4})(0 \leq r \leq n - 4)$ is $n - 3$
. . .
. . .
. . .
Number of triplets, $\left(3^r, 3^{r+\frac{n-1}{2}}, 3^{r+n-1}\right)$ (n odd) is 2
Number of triplets, $\left(3^r, 3^{r+\frac{n}{2}}, 3^{r+n}\right)$ (n even) is 1
$\therefore$ If n is odd, the number of favourable outcomes
$=(n-1)+(n-3)+....+4+2=\frac{n^2-1}{4}$
and if n is even, the number of favourable outcomes
$=(n-1)+(n-3)+....+3+1=\frac{n}{2}\times\frac{n}{2}=\frac{n^2}{4}$
$\therefore$ Required Probability $=\frac{(n^2-1)/4}{^{(n+1)}C_3}=\frac{3}{2n}$, if n is odd

or $\dfrac{n^2/4}{{}^{(n+1)}C_3} = \dfrac{3n}{2(n^2-1)}$ if n is even

4. (b) : $\dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{e^{\tan^{-1} y}}{1+y^2}$

$\text{I.F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

$\therefore$ General solution is $x \cdot e^{\tan^{-1} y} = \int \dfrac{\left(e^{\tan^{-1} y}\right)^2}{1+y^2} dy$

$\Rightarrow \quad x.e^{\tan^{-1} y} = \dfrac{e^{2\tan^{-1} y}}{2} + c_1$

$\Rightarrow \quad 2xe^{\tan^{-1} y} = e^{2\tan^{-1} y} + c \quad \therefore \ f(y) = \tan^{-1} y$

$\therefore$ Area $= \int_{-\sqrt{3}}^{\sqrt{3}} |\tan^{-1} y| dy = 2\int_0^{\sqrt{3}} \tan^{-1} y dy$

$= 2\left[(y\tan^{-1} y)_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \dfrac{y}{1+y^2} dy \right]$

$= 2\dfrac{\pi}{\sqrt{3}} - \left[\log(1+y^2)\right]_0^{\sqrt{3}} = \dfrac{2\pi}{\sqrt{3}} - \log 4$

5. (b, c) :

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{\tan^2 x}{(x^2-[x]^2)} = \lim\limits_{x\to 0^+} \dfrac{\tan^2 x}{x^2} = 1$

As $x \to 0^+$, $[x] = 0$

Also, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \sqrt{(x-[x])\cot(x-[x])} = \sqrt{\cot 1}$

$\cot^{-1}\left(\lim\limits_{x\to 0^-} f(x)\right)^2 = \cot^{-1}(\cot 1) = 1$

6. (a, c) : Let $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

$\therefore \ |\vec{a}| = 3, |\vec{b}| = 3$

$\therefore$ Vector along bisectors is given by

$= \dfrac{\vec{a}}{|\vec{a}|} \pm \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{2\hat{i}-2\hat{j}+\hat{k}}{3} \pm \dfrac{\hat{i}+2\hat{j}-2\hat{k}}{3}$

$= \hat{i} - \dfrac{1}{3}\hat{k}, \dfrac{1}{3}\hat{i} - \dfrac{4}{3}\hat{j} + \hat{k}$

$\therefore$ Required vectors are

$= 2.\dfrac{\hat{i}-\frac{1}{3}\hat{k}}{\sqrt{1^2+\left(-\frac{1}{3}\right)^2}}, 2.\dfrac{\frac{1}{3}\hat{i}-\frac{4}{3}\hat{j}+\hat{k}}{\sqrt{\left(\frac{1}{3}\right)^2+\left(-\frac{4}{3}\right)^2+1^2}}$

7. (b, d) : Domain of $f = R$, Domain $g = R - [-1, 0)$; because $-1 \le x < 0$ thus $1 + [x] = 0$

$\therefore$ Domain of $f - g = R - [-1, 0)$
since $e^{-x} > 0 \Rightarrow (1 + [x])y > 0$
Either $y > 0 \Rightarrow 1 + [x] > 0$, or $y < 0 \Rightarrow 1 + [x] < 0$
$\therefore \ y \in R - \{0\}$

8. (8) : The 8 vertices of the block gives ${}^8C_3 = 56$ triangles, each of which is either acute or right angled. Each vertex of the block serves as the vertex of the right angle in three triangles whose hypotenuse are face diagonals of the block, and three triangles whose hypotenuse are space diagonals of the block. Hence there are $8(3 + 3) = 48$ right triangles and so $56 - 48 = 8$ acute triangles.

9. (6) : The circles are given as $x^2 + y^2 = 12$ and $x^2 + y^2 - 5x + 3y - 2 = 0$

Common chord AB is $5x - 3y - 10 = 0$. Let the coordinates of P be (α, β)

Equation of the chord of contact of $F(\alpha, \beta)$ with respect to $x^2 + y^2 = 12$ is $x\alpha + y\beta - 12 = 0$ comparing the coefficients of common chord AB and chord of contact is $\dfrac{\alpha}{5} = \dfrac{\beta}{-3} = \dfrac{-12}{-10} \Rightarrow \alpha = 6 \quad \therefore$ x-coordinate is 6.

10. (2) :

$\dfrac{1}{\cos a_1 \cos a_2} + \dfrac{1}{\cos a_2 \cos a_3} + \ldots\ldots + \dfrac{1}{\cos a_{n-1} \cos a_n}$

$\Rightarrow \dfrac{1}{\sin d}\left(\dfrac{\sin(a_2-a_1)}{\cos a_1 \cos a_2} + \dfrac{\sin(a_3-a_2)}{\cos a_2 \cos a_3} + \ldots + \dfrac{\sin(a_n-a_{n-1})}{\cos a_{n-1}\cos a_n}\right)$

$= \dfrac{1}{\sin d}((\tan a_2 - \tan a_1) + (\tan a_3 - \tan a_2) + \ldots + (\tan a_n - \tan a_{n-1}))$

$= \dfrac{1}{\sin d}(\tan a_n - \tan a_1) \Rightarrow k = \dfrac{1}{\sin d} = 2$

11. (1) : The function $y = f(x) = x^2 - x + 1$ $= \left(x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$ increases in the interval $\left[\dfrac{1}{2}, \infty\right)$ and x varying in the indicated interval we have

$y = f(x) \ge \dfrac{3}{4}$ i.e., $y \in \left[\dfrac{3}{4}, \infty\right)$

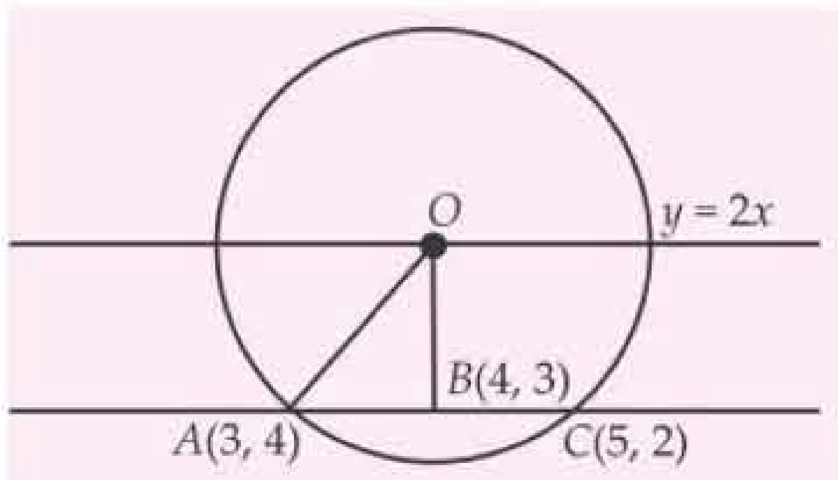
12. (6) : $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f(x)f(h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x)[1+hg(h)H(h)-1]}{h}$

$= f(x)\lim\limits_{h\to 0} g(h)\cdot H(h) = 6f(x)$

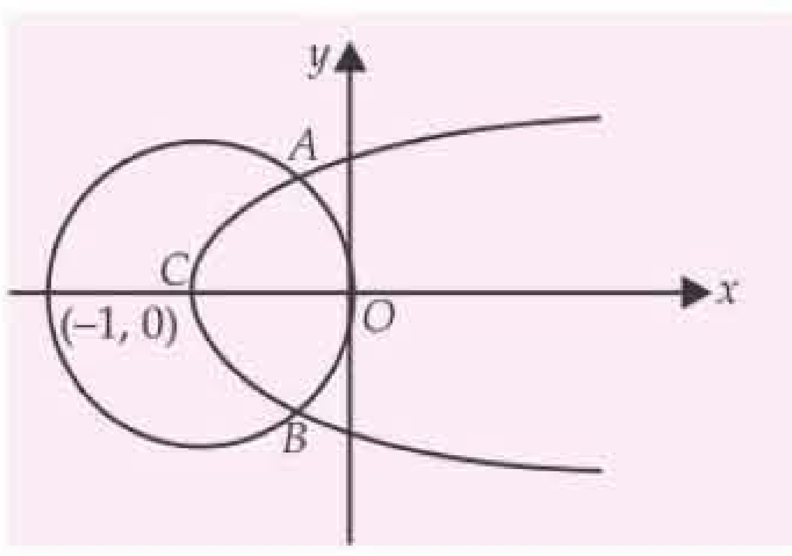
13. A – q; B – r; C – p

(A) Equation of perpendicular bisector of side AC is $x - y = 1$



So, centre O is $(-1,-2)$; Radius $= \sqrt{52}$

(B)



Point of intersection of the two curves are

$$A\left(\frac{-1}{2},\frac{\sqrt{3}}{2}\right) \text{ and } B\left(\frac{-1}{2},-\frac{\sqrt{3}}{2}\right)$$

So, common chord length $AB=\sqrt{3}$.

(C) $P_1P_2 = b^2,\ \dfrac{x^2}{4}+\dfrac{y^2}{25}=1,$ so $P_1P_2 = 4$

14. A → s; B → r; C → r; D → q

(A) $I_1 = \dfrac{4!.5!}{10!},\ I_2 = \dfrac{5!.5!}{11!} \Rightarrow \dfrac{I_2}{I_1} = \dfrac{5}{11}$

(B) $I_1 = \dfrac{3.5.3}{10.8.6.4.2}.\pi,\ I_2 = \dfrac{3.3}{8.6.4.2}2\pi \Rightarrow \dfrac{I_1}{I_2} = \dfrac{1}{4}$

(C) $\displaystyle\int_{-a}^{a} f(x)dx = \int_0^a (f(x)+f(-x))dx$

$$\therefore\ I = \int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\sin^2 2x}\left(\frac{1}{1-e^{-3x}}+\frac{1}{1-e^{3x}}\right)dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \sec^2 x dx = \frac{1}{4}$$

(D) $\tan x = t \Rightarrow I = \displaystyle\int_0^{\infty} \frac{dt}{(\sqrt{t}+1)^4}$

$\sqrt{t} = u-1$

$$\Rightarrow\ I = 2\int_1^{\infty}\frac{u-1}{u^4}du = 2\int_1^{\infty}(u^{-3}-u^{-4})du = \frac{1}{3}$$

15. A - r; B - p; C - r; D - q

(A) $y = |\cos x|,\ y = 2[x]$

$\therefore$ Number of solutions will be 0.

(B) The number of solutions is 8

(C) $\sin^3 x\cos x + \sin^2 x\cos^2 x + \sin x\cos^3 x = 1$

$\Rightarrow \sin x\cos x(\sin^2 x + \sin x\cos x + \cos^2 x) = 1$

$$\Rightarrow \frac{\sin 2x}{2}\left(1+\frac{\sin 2x}{2}\right)=1$$

$\Rightarrow \sin 2x(2 + \sin 2x) = 4 \Rightarrow \sin^2 2x + 2\sin 2x - 4 = 0$

$$\Rightarrow \sin 2x = \frac{-2\pm\sqrt{4+16}}{2} = -1\pm\sqrt{5} \text{ (Impossible)}$$

(D) $\dfrac{2\pi}{|k|} = \dfrac{\pi}{2} \Rightarrow |k| = 4$

PAPER-II

1. (a) : Equation of the plane through $P(h, k, l)$ perpendicular to OP is

$xh + yk + zl = h^2 + k^2 + l^2 = p^2$; where, $p^2 = h^2 + k^2 + l^2$

$$\Rightarrow \frac{x}{\frac{p^2}{h}}+\frac{y}{\frac{p^2}{k}}+\frac{z}{\frac{p^2}{l}}=1$$

$$\Delta_{xy} = \frac{1}{2}\cdot\frac{p^2}{h}\cdot\frac{p^2}{k},\ \Delta_{yz} = \frac{1}{2}\cdot\frac{p^2}{k}\cdot\frac{p^2}{l},\ \Delta_{zx} = \frac{1}{2}\cdot\frac{p^2}{l}\cdot\frac{p^2}{h}$$

$$A = \sqrt{(\Delta_{xy})^2+(\Delta_{yz})^2+(\Delta_{zx})^2} = \frac{p^4}{2}\sqrt{\frac{l^2+h^2+k^2}{h^2k^2l^2}}$$

$$= \frac{p^4}{2}\sqrt{\frac{p^2}{h^2k^2l^2}} = \frac{p^5}{2hkl}$$

Hence, $Ar\ (\Delta ABC) = \dfrac{p^5}{2hkl}$

2. (c) : Let $E_1 : s_1$ and s_2 are in the same group

$E_2 : s_1$ and s_2 are in the different group

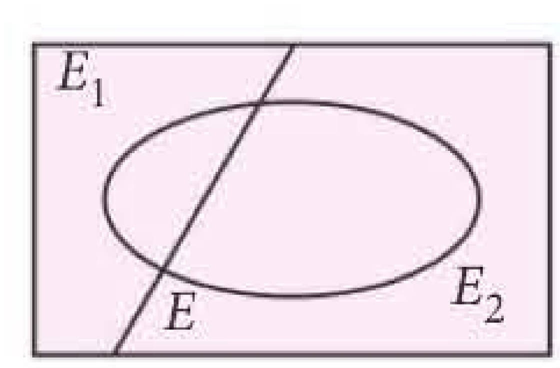
E : exactly one of the two players s_1 and s_2 is among the eight winners.

$E = (E \cap E_1) \cup (E \cap E_2)$

$\Rightarrow P(E) = P(E \cap E_1) + P(E \cap E_2)$

$\Rightarrow P(E) = P(E_1)\cdot P(E/E_1) + P(E_2)\cdot P(E/E_2)$

Now $P(E_1) = \dfrac{\dfrac{(14)!}{(2)^7\cdot 7!}}{\dfrac{16!}{2^8\cdot 8!}} = \dfrac{1}{15}$



$P(E_2) = 1 - \dfrac{1}{15} = \dfrac{14}{15}$

Hence $P(E) = \dfrac{1}{15}\cdot 1 + \dfrac{14}{15}\cdot P$ (exactly one of either S_1 or S_2 wins)

$$= \frac{1}{15}+\frac{14}{15}\cdot\left(\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\right) = \frac{1}{15}+\frac{14}{15}\cdot\frac{1}{2} = \frac{1}{15}+\frac{7}{15} = \frac{8}{15}.$$

3. (b): Put $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2,\ \tan\theta = \dfrac{y}{x}$

$\Rightarrow xdx + ydy = rdr$ and $\dfrac{x\,dy - y\,dx}{x^2} = \sec^2\theta\,d\theta$

$\therefore$ Given equation $\Rightarrow \dfrac{r\,dr}{r^2 d\theta} = \sqrt{\dfrac{a^2 - r^2}{r^2}}$

$\Rightarrow \dfrac{dr}{\sqrt{a^2 - r^2}} = d\theta \Rightarrow \sin^{-1}\left(\dfrac{r}{a}\right) = \theta + c$

$\Rightarrow r = a\sin(\theta + c)$

4. (b): Number of students who gave wrong answers to atleast i questions $= 2^{n-i}$

Number of students who gave wrong answers to atleast n questions $= 2^0 = 1$

Number of students gave wrong answers to exactly i questions $= 2^{n-i} - 2^{n-(i+1)}$

Number of wrong answers

$= \sum_{i=1}^{n-1} i\left(2^{n-i} - 2^{n-(i+1)}\right) + (n) = 2047$

$\Rightarrow 1 + 2^1 + 2^2 + \ldots + 2^{n-1} = 2047$

$\Rightarrow 2^n = 2048 \Rightarrow n = 11$

5. (a): $BC = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \Rightarrow BC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore tr(A) + tr\left(\dfrac{A}{2}\right) + tr\left(\dfrac{A}{2^2}\right) + \ldots\ldots$

$= tr(A) + \dfrac{1}{2}tr(A) + \dfrac{1}{2^2}tr(A) + \ldots$

$= \dfrac{tr(A)}{1 - (1/2)} = 2tr(A) = 2(2+1) = 6$

6. (d): Since a and b are the roots of $x^2 - 10cx - 11d = 0$, we have

(i) $a + b = 10c$ and (ii) $ab = -11d$ ... (1)

Also, since c and d are the roots of $x^2 - 10ax - 11b = 0$, we have

(i) $c + d = 10a$ and (ii) $cd = -11b$ ... (2)

Adding part (i) of Eqs. (1) and (2), we get

$a + b + c + d = 10(a + c)$

$\Rightarrow b + d = 9(a + c)$ ... (3)

Multiplying part (ii) of Eqs. (1) and (2), we get

$abcd = 121bd \Rightarrow ac = 121$ ... (4)

Also, $a^2 - 10ca - 11d = 0 = c^2 - 10ca - 11b$

$\Rightarrow a^2 + c^2 - 20ca - 11(b + d) = 0$

From Eqs. (3) and (4), we have

$a^2 + c^2 - 20(121) - 99(a + c) = 0$

$\Rightarrow (a + c)^2 - 2 \times 121 - 20 \times 121 - 99(a + c) = 0$

$\Rightarrow (a + c - 121)(a + c + 22) = 0$

$\Rightarrow a + c = 121\ ;\ a + c = -22$

Since a, c are positive, $a + c \neq -22$.

Therefore $a + c = 121$ and

$a + b + c + d = (a + c) + 9(a + c) = 1210$

7. (c): $f_3 \cdot f_2 \cdot f_1(x, y) = f_3 \cdot f_2(y, x) = f_3(y + 3x, x)$

$= \left(\dfrac{y + 3x - x}{2}, \dfrac{y + 3x + x}{2}\right) = \left(\dfrac{2x + y}{2}, \dfrac{4x + y}{2}\right)$

$\therefore$ $A(0, 0) \to A'(0,0)$, $B(2, 0) \to B'(2, 4)$, $C(2, 2) \to C'(3, 5)$, $D(0, 2) \to D'(1, 1)$

$\therefore$ It is easily seen $A'B' = D'C'$, $A'D' = B'C'$, $A'B' \neq B'C'$.

Hence, it is neither a square nor a rhombus.

Further slope of $A'B' \times A'D' = \dfrac{4}{2}\cdot\dfrac{1}{1} \neq -1$

Hence AB is not perpendicular to AD.

$\Rightarrow$ It is not a rectangle

$\therefore$ $ABCD$ is a parallelogram.

8. (a, b, c): $(\cos x + \cos y)^2 = a^2$

$\Rightarrow \cos^2 x + \cos^2 y + 2\cos x \cos y = a^2$ ... (1)

$\cos 2x + \cos 2y = b$

$\Rightarrow 2\cos^2 x - 1 + 2\cos^2 y - 1 = b$

$\Rightarrow 2[\cos^2 x + \cos^2 y] = b + 2$

$\Rightarrow \cos^2 x + \cos^2 y = \dfrac{b}{2} + 1$ ... (2)

From (1) and (2),

$2\cos x \cos y = a^2 - \left(\dfrac{b+2}{2}\right)$

$\Rightarrow \cos x \cos y = \dfrac{a^2}{2} - \left(\dfrac{b+2}{4}\right)$

$\cos 3x + \cos 3y = c$

$\Rightarrow 4\cos^3 x - 3\cos x + 4\cos^3 y - 3\cos y = c$

$\Rightarrow 4[\cos^3 x + \cos^3 y] - 3[\cos x + \cos y] = c$

$\Rightarrow 4[(\cos x + \cos y)(\cos^2 x + \cos^2 y - \cos x \cos y)] - 3(\cos x + \cos y) = c$

$\Rightarrow 4\left[a\left(\dfrac{b+2}{2} - \dfrac{1}{2}\left(a^2 - \dfrac{b+2}{2}\right)\right)\right] - 3a = c$

$\Rightarrow 2ab + 4a - 2a^3 + ab + 2a = 3a + c$

$\Rightarrow 2a^3 + c = 3a(1 + b)$

9. (a,d): $f'(x) = \dfrac{2x}{(\log x^2)^2} - \dfrac{1}{(\log x)^2} = \dfrac{1}{(\log x)^2}\left[\dfrac{x}{2} - 1\right]$

$\Rightarrow f'(x) > 0 \Rightarrow x \in (2, \infty) \Rightarrow f'(x) < 0 \Rightarrow x \in (0, 2)$

10. (a, b): Let $A + B = f(x)$

$= \int_{e^{-1}}^{\tan x} \dfrac{t\,dt}{t^2 + 1} + \int_{e^{-1}}^{\cot x} \dfrac{dt}{t(1 + t^2)}$

$$\Rightarrow f'(x)=\frac{\tan x}{\tan^2 x+1}\cdot\sec^2 x+\frac{1}{\cot x(1+\cot^2 x)}(-\text{cosec}^2 x)=0$$

$$\Rightarrow f(x)\text{ is constant } \Rightarrow f(x)=f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow f(x)=\int_{e^{-1}}^{1}\frac{tdt}{t^2+1}+\int_{e^{-1}}^{1}\frac{dt}{t(t^2+1)}=\int_{e^{-1}}^{1}\frac{dt}{t}=1$$

$$\Rightarrow f(x)=1 \text{ for all } x \text{ in}\left(0,\frac{\pi}{2}\right)$$

11. (a, b, c) : $h'(x)=2x\{f'(x^2/3)-f'(3-x^2)\}$

$f'(x^2/3)>f'(3-x^2)\,\forall x$ such that

$$\frac{x^2}{3}>3-x^2 \Rightarrow x^2>\frac{9}{4}$$

$f'(x^2/3)<f'(3-x^2)\ \forall x$ such that $x^2<9/4$

$h(x)$ increases in $(-3/2, 0)\cup(3/2, 4)$ and $h(x)$ decreases in $(0, 3/2)$

12. (a,b,c) : Equation of chord PQ is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

This passes through focus $(ae, 0)$

$$\therefore\quad e=\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)} \Rightarrow \frac{e+1}{e-1}=\cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2}$$

$$\Rightarrow \tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right)+\frac{1-e}{e+1}=0$$

13. (b, c, d) : Slope of normal, $\tan\theta=-\dfrac{dx}{dy}$

$\therefore$ The given equation becomes at a general point (x, y)

$$y\left(-x^2\left(-\frac{dx}{dy}\right)+\frac{dy}{dx}\right)=x(y^2+1)$$

$$\Rightarrow yx^2+y\left(\frac{dy}{dx}\right)^2=\frac{dy}{dx}.x(y^2+1)$$

$$\Rightarrow y\left(\frac{dy}{dx}\right)^2-x(y^2+1)\frac{dy}{dx}+yx^2=0$$

$$\Rightarrow yy'^2-xy^2y'-xy'+yx^2=0$$

$$\Rightarrow yy'(y'-xy)-x(y'-xy)=0$$

$$\therefore\ \frac{dy}{dx}=xy \text{ or } \frac{dy}{dx}=\frac{x}{y}$$

$$\Rightarrow \log y=\frac{x^2}{2}+c \text{ or } x^2-y^2=c$$

$$\Rightarrow\ y=ke^{\frac{x^2}{2}} \text{ or } \log y^2=x^2-\log k$$

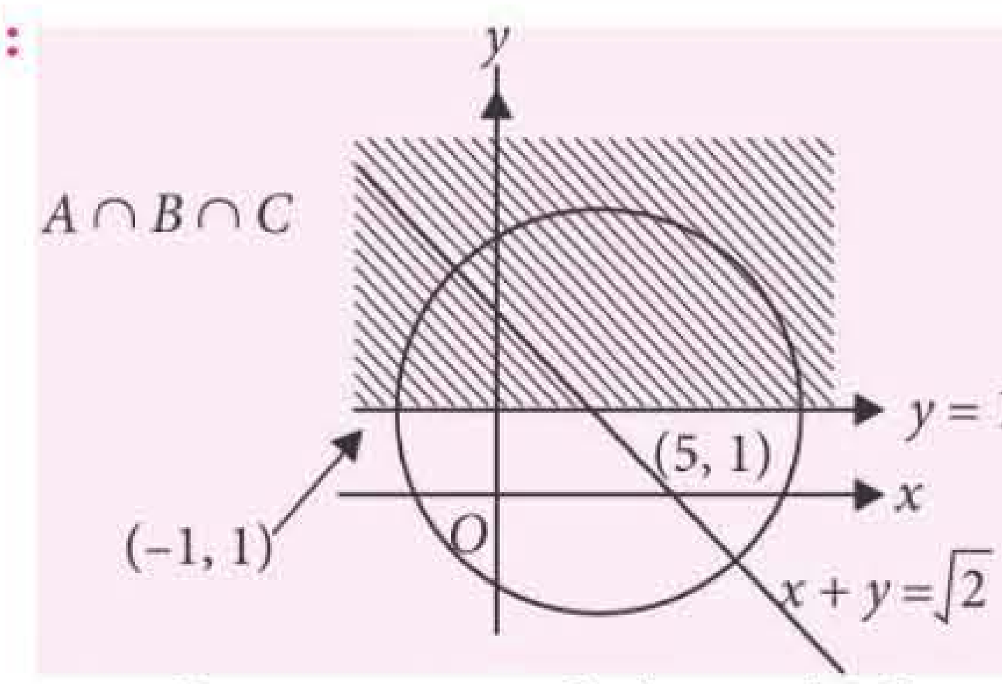
$\Rightarrow \log(ky^2)=x^2$

14. (a, b, c, d) :
$$\sum_{r=1}^{n}S_r=\begin{vmatrix}\sum_{r=1}^{n}2r & x & n(n+1)\\ \sum_{r=1}^{n}(6r^2-1) & y & n^2(2n+3)\\ \sum_{r=1}^{n}(4r^3-2nr) & z & n^3(n+1)\end{vmatrix}$$

$$=\begin{vmatrix}n(n+1) & x & n(n+1)\\ n^2(2n+3) & y & n^2(2n+3)\\ n^3(n+1) & z & n^3(n+1)\end{vmatrix}=0$$

(as C_1 and C_3 are same)

15. (b):



A is the set of points on and above the line $y = 1$ ...(i) in the Argand plane. B is the set of points on the circle $(x-2)^2+(y-1)^2=9$ ...(ii) and

$C=\text{Re}(1-i)z=\text{Re}(1-i)(x+iy)$

$\Rightarrow\ x+y=\sqrt{2}$ ... (iii)

Hence $A\cap B\cap C$ has only one point of intersection.

16. (d): $|z|-|w|<|z-w|$ and $|z-w|$ is the distance between z and w. Here, z is fixed. Hence distance between z and w would be maximum for diametrically opposite points. Therefore,

$|z-w|<6 \Rightarrow -6<|z|-|w|<6$

$\Rightarrow -3<|z|-|w|+3<9$

17. (c) : $\phi(1)=1$, $\phi(p^n)=p^{n-1}(p-1)$

$\phi(mn)=\phi(m)\cdot\phi(n)$

$\phi(8n+4)=\phi(4(2n+1))=\phi(4)\cdot\phi(2n+1)$

$=\phi(2)^2\cdot\phi(2n+1)=2\cdot\phi(2n+1)$

18. (b): $\phi(n)$ is odd $\Rightarrow$ $\phi(p^n)$ is odd

$\Rightarrow$ $p^{n-1}(p-1)$ is odd.

$\because$ p is prime. The only value p can take is $p=2$

$\therefore$ $\phi(2^n)$ is odd.

$\Rightarrow$ $2^{n-1}(2-1)=2^{n-1}$ is odd.

$\Rightarrow$ $n-1=0\Rightarrow n=1$

$\therefore$ $\phi(1)=1=\phi(2)$

CONCEPT MAP

SEQUENCES AND SERIES

Class XI

Sequence
A sequence is a function from natural number N (domain) to real numbers (codomain)

Series
If $a_1, a_2, ..., a_n$ is a sequence, then the expansion $a_1 + a_2 + ... + a_n$ is called the series.

Progression
If the terms of a sequence are written under specific conditions, then the sequence is called progression.

Types of Progression

Sum of n terms of Special Series

- Sum of n natural numbers $\sum n = \frac{n(n+1)}{2}$
- Sum of squares of n natural numbers $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$
- Sum of cubes of n natural numbers $\sum n^3 = \frac{n^2(n+1)^2}{4} = \left(\sum n\right)^2$
- $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$
- $1\cdot 2 + 2\cdot 3 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}$

Arithmetic Progression (A.P.)

A sequence whose terms increases or decreases by a fixed number.

n^{th} term : $T_n = a + (n-1)d$
where d (common difference) $= T_n - T_{n-1}$, a = first term,

n^{th} term from end
$T_n' = l - (n-1)d$, where
l = last term

Sum of n terms

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l]$$

Note
- 3 terms in an A.P. $a - d, a, a + d, a + 2d$
- 4 terms in an A.P. $a - 2d, a - d, a, a + d, a + 2d$

Basic Properties
- If $a_1, a_2, a_3, ..., a_n$ are in A.P. with common difference d, then
  - $a_1 \pm x, a_2 \pm x, ..., a_n \pm x$ are also in A.P. where x is constant.
  - $ka_1, ka_2, ..., ka_n$ are also in A.P. with common difference kd.
  - $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, ..., \frac{a_n}{k}$ are also in A.P. with common difference, $\frac{d}{k}, k \uparrow 0$.

Geometric Progression (G.P.)

A sequence of non-zero numbers for which the ratio of a term to its just preceding term is always constant.

n^{th} term : $T_n = ar^{n-1}$, where r(common ratio) $= T_n / T_{n-1}$, a = first term

n^{th} term from end : $T_n' = l/r^{n-1}$, l = last term

Sum of n terms

$$S_n = \begin{cases} \frac{a(r^n - 1)}{r - 1}, r > 1 \\ \frac{a(1 - r^n)}{1 - r}, r < 1 \\ an, \quad r = 1 \end{cases} ; S_\infty = \frac{a}{1-r}, |r| < 1$$

Note
- 3 terms in G.P. $a/r, a, ar$.
- 4 terms in G.P. $a/r^2, a/r, a, ar$.

Basic Properties
- If $a_1, a_2, a_3, ..., a_n$ are in G.P., with common ratio r then
  - $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ..., \frac{1}{a_n}$ also in G.P., with common ratio $1/r$.
  - $ka_1, ka_2, ka_3, ..., ka_n$ or $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, ...$ $(k \neq 0)$ also in G.P., with common ratio r.
  - $a_1^n, a_2^n, a_3^n, ..., a_n^n$ also in G.P. with common ratio r^n.
  - $a_1 a_n = a_k \cdot a_{n-k+1}$ $\forall k = 1, 2, 3, ..., n-1$

Means

Arithmetic mean (A.M.)
- For two numbers a and b, A.M. is $\frac{a+b}{2}$.
- $A_r = a + r\left(\frac{b-a}{n+1}\right), \forall r = 1, 2, ..., n$
  where $A_1, ..., A_n$ are n arithmetic means inserted between two numbers a and b.

Geometric mean (G.M.)
- For two numbers a and b, G.M. is $\sqrt{ab}$.
- $G_k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}} \forall k = 1, 2, 3, ..., n$
  where $G_1, ... , G_n$ are n geometric means inserted between two numbers a and b.

DEFINITE INTEGRALS AND APPLICATION OF INTEGRALS

CONCEPT MAP

Class XII

DEFINITE INTEGRALS

For any two values a and b, we have $\int_a^b f(x)\,dx = [F(x) + c]_a^b = F(b) - F(a)$

Limit of Sum

$\int_a^b f(x)\,dx = \lim_{h \to 0} h[f(a) + f(a+h) + ... + f(a + (n-1)h)]$,
where $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

Fundamental Theorem of Calculus

- **First Fundamental Theorem :** Let $f(x)$ be a continuous function in the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$.
- **Second Fundamental Theorem :** Let $f(x)$ be a continuous function in the closed interval $[a, b]$ and $F(x)$ be an integral of $f(x)$, then $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

Solving by Substitution

When definite integral is to be found by substitution, change the lower and upper limits of integration. If substitution is $t = f(x)$ and lower limit of integration is a and upper limit is b, then new lower and upper limits will be $f(a)$ and $f(b)$ respectively.

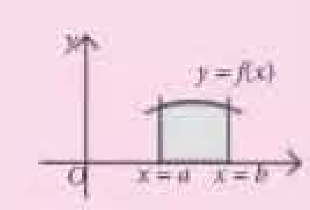
Properties

- $\int_a^b f(x)dx = \int_a^b f(t)dt$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$. In particular $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a < c < b$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ • $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- $\int_{-a}^a f(x)dx = \begin{cases} 0 & ,\text{if } f(-x) = -f(x) \\ 2\int_0^a f(x)dx & ,\text{if } f(-x) = f(x) \end{cases}$
- $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$
- $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

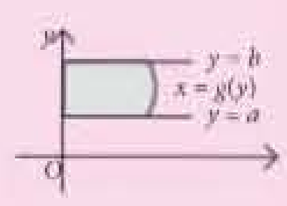
APPLICATION OF INTEGRALS

Area Under Simple Curves

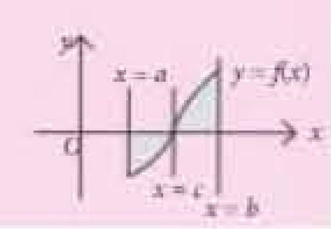
- Area $= \int_a^b y\,dx$
  $= \int_a^b f(x)dx$ (where $b > a$)



- Area $= \int_a^b x\,dy$
  $= \int_a^b g(y)dy$ (where $b > a$)

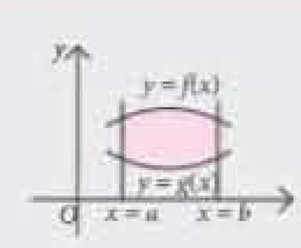


- Area $= \left|\int_a^c f(x)dx\right| + \int_c^b f(x)dx$

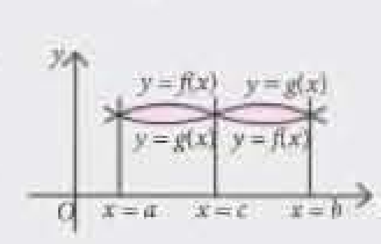


Area Between Two Curves

- Area $= \int_a^b [f(x) - g(x)]dx$,
  $f(x) \geq g(x)$ in $[a, b]$

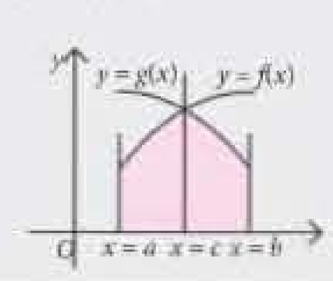


- Area $= \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$



where $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$

- Area $= \int_a^c f(x)dx + \int_c^b g(x)dx$



JEE MAIN 2019

Between 6th to 20th January | Between 6th to 20th April

MOCK DRILL FOR JEE

* ALOK KUMAR

1. If in a triangle ABC, $\cos 3A + \cos 3B + \cos 3C = 1$, then one angle must be exactly equal to

(a) $\pi/3$ (b) $2\pi/3$ (c) π (d) $\pi/6$

2. The value of $\sum_{r=0}^{10} \cos^3 \frac{\pi r}{3}$ is equal to

(a) $\frac{-9}{2}$ (b) $\frac{-7}{2}$ (c) $\frac{-9}{8}$ (d) $\frac{-1}{8}$

3. The value of $\sin^3 10° + \sin^3 50° - \sin^3 70°$ is equal to

(a) $-\frac{3}{2}$ (b) $\frac{3}{4}$ (c) $-\frac{3}{4}$ (d) $-\frac{3}{8}$

4. Sum of integral values of n such that $\sin x\ (2\sin x + \cos x) = n$, has at least one real solution is

(a) 3 (b) 1 (c) 2 (d) 0

5. If α, β, γ are the roots of the equation $x^3 + px + q = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is

(a) 4 (b) 2 (c) 0 (d) -2

6. One vertex of an equilateral triangle is at the origin and the other two vertices are roots of $2z^2 + 2z + k = 0$, then k is

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{2}$

7. If the arithmetic mean of two positive numbers a and b $(a > b)$ is twice their G.M., then $a : b$ is

(a) $6+\sqrt{7} : 6-\sqrt{7}$ (b) $2+\sqrt{3} : 2-\sqrt{3}$

(c) $5+\sqrt{6} : 5-\sqrt{6}$ (d) $4+\sqrt{3} : 4-\sqrt{3}$.

8. If $\sum_{r=0}^{n} \left\{ \frac{{}^nC_{r-1}}{{}^nC_r + {}^nC_{r-1}} \right\}^3 = \frac{25}{24}$, then n is equal to

(a) 3 (b) 4

(c) 5 (d) 6

9. $f(x) = \text{Max}\{\sin x, \cos x\}\ \forall\ x \in R$ then range of $f(x)$ is

(a) $\left[\frac{-1}{\sqrt{2}}, 1\right]$ (b) $\left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

(c) $[-1, 1]$ (d) ϕ

10. If $f(x) = \sqrt{\cos(\sin x)} + \sqrt{\sin(\cos x)}$, then the range of $f(x)$ is

(a) $[\sqrt{\cos 1}, \sqrt{\sin 1}]$ (b) $[\sqrt{\cos 1}, 1 + \sqrt{\sin 1}]$

(c) $[1 - \sqrt{\cos 1}, \sqrt{\sin 1}]$ (d) $[\sqrt{\cos 1}, 1]$

11. For each positive integer n, let

$$s_n = \frac{3}{1.2.4} + \frac{4}{2.3.5} + \frac{5}{3.4.6} + \ldots.. + \frac{n+2}{n(n+1)(n+3)}$$

Then $\lim_{n\to\infty} s_n$ equals

(a) $\frac{29}{6}$ (b) $\frac{29}{36}$ (c) 0 (d) $\frac{29}{18}$

12. $\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n}$ equals

(a) e (b) e^{-1}

(c) e^{-2} (d) e^2

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13. Let 'f' be a real valued function defined on the interval $(-1, 1)$ such that $e^{-x} \cdot f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt$ $\forall\, x \in (-1, 1)$ and let 'g' be the inverse function of 'f'. Then $g'(2)$ equals

(a) 3 (b) 1/2 (c) 1/3 (d) 2

14. Let $f: R \to R$ be a differentiable function satisfying $f(y)f(x-y) = f(x)\ \forall\, x, y \in R$ and $f'(0) = p, f'(5) = q$, then $f(5)$ equals

(a) p^2/q (b) p/q (c) q/p (d) q

15. The domain of the derivative of the function

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases} \text{ is}$$

(a) $R - \{0\}$ (b) $R - \{1\}$
(c) $R - \{-1\}$ (d) $R - \{-1, 1\}$

16. M is the mid point of side AB of equilateral triangle ABC. P is a point on BC such that $AP + PM$ is minimum. If $AB = 20$ units then $AP + PM$ is

(a) $10\sqrt{7}$ (b) $10\sqrt{3}$ (c) $10\sqrt{5}$ (d) 10

17. The orthocentre of the triangle formed by the lines AB, AC and BC given by $x + y = 1$, $2x + 3y = 6$ and $4x - y + 4 = 0$ respectively lies in

(a) I quadrant (b) II quadrant
(c) III quadrant (d) IV quadrant

18. The complete set of values of 'a' for which the point (a, a^2), $a \in R$ lies inside the triangle formed by the lines $x - y + 2 = 0$, $x + y = 2$ and x-axis is

(a) $(-2, 2)$ (b) $(-1, 1) - \{0\}$
(c) $(0, 2)$ (d) $(-2, 0)$

19. Equation of circle touching the line $|x - 2| + |y - 3| = 4$ will be

(a) $(x-2)^2 + (y-3)^2 = 12$
(b) $(x-2)^2 + (y-3)^2 = 4$
(c) $(x-2)^2 + (y-3)^2 = 10$
(d) $(x-2)^2 + (y-3)^2 = 8$

20. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where $(pq \ne 0)$) are bisected by the x-axis, then

(a) $p^2 = q^2$ (b) $p^2 = 8q^2$
(c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

21. If α is a root of $x^4 = 1$ with negative principal argument, then the principal argument of $\Delta(\alpha)$ where

$$\Delta(\alpha) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha^n & \alpha^{n+1} & \alpha^{n+3} \\ \frac{1}{\alpha^{n+1}} & \frac{1}{\alpha^n} & 0 \end{vmatrix} \text{ is}$$

(a) $5\pi/4$ (b) $-3\pi/4$ (c) $\pi/4$ (d) $-\pi/4$

22. If $\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ac+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$ $= (1 + a^2 + b^2 + c^2)^3$,

then λ is equal to

(a) 0 (b) 1 (c) -1 (d) ± 1

23. If the area of the rhombus enclosed by the lines $lx \pm my \pm n = 0$ be 2 square units, then

(a) $l, 2m, n$ are in G.P (b) l, n, m are in G.P
(c) $lm = n$ (d) $ln = m$

24. An equilateral triangle has its centroid at origin and one side is $x + y = 1$. The equations of the other sides are

(a) $y+1=(2\pm\sqrt{3})(x+1)$
(b) $y+1=(2\pm\sqrt{3})x,\ y+1=(3\pm\sqrt{3})x$
(c) $y+1=(3\pm\sqrt{3})(x-1),\ y+1=\sqrt{3}x$
(d) $y\pm1=(3\pm\sqrt{3})(x-1),\ y+1=\frac{\sqrt{3}-1}{\sqrt{3}+1}(x+1)$

25. If $f(x)=x^2+x+\frac{3}{4}$ and $g(x) = x^2 + ax + 1$ be two real functions, then the range of a for which $g(f(x)) = 0$ has no real solution is

(a) $(-\infty, -2)$ (b) $(-2, 2)$
(c) $(-2, \infty)$ (d) $(2, \infty)$

26. $\lim_{\theta\to 0}\left\{\left[\frac{n\sin\theta}{\theta}\right]+\left[\frac{n\tan\theta}{\theta}\right]\right\} =$ ____ where $[x]$ is greatest integer $\le x$ and $n \in I$

(a) $2n$ (b) $2n + 1$
(c) $2n - 1$ (d) 0

27. If the equation $\sin^2 x - a \sin x + b = 0$ has only one solution in $(0, \pi)$, then which of the following statements is correct ?

(a) $a \in (-\infty, 1] \cup (2, \infty)$ (b) $b \in [1, \infty)$
(c) $a = 2 + b$ (d) None of these

28. If area of triangle formed by tangents from the point (x_1, y_1) to the parabola $y^2 = 4ax$ and their chord of contact is

(a) $\dfrac{(y_1^2-4ax_1)^{3/2}}{2a^2}$ (b) $\dfrac{(y_1^2-4ax_1)^{3/3}}{a^2}$

(c) $\dfrac{(y_1^2-4ax_1)^{3/2}}{2a}$ (d) none of these

29. If $n \ge 3$ and $1, \alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n-1}$ are n roots of unity, then value of $\sum_{1\le i<j\le n-1} \alpha_i\alpha_j$ is

(a) 0 (b) 1 (c) -1 (d) $(-1)^n$

30. If $C_0, C_1, C_2, ..., C_n$ are the binomial coefficients in the expansion $(1+x)^n$, n being even, then $C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + ... + (C_0 + C_1 + C_2 + ... + C_{n-1})$ is equal to

(a) $n\,2^n$ (b) $n \cdot 2^{n-1}$

(c) $n \cdot 2^{n-1}$ (d) $n \cdot 2^{n-2}$

SOLUTIONS

1. (b) : Given, $\cos 3A + \cos 3B + \cos 3C = 1$

$\Rightarrow \cos 3A + \cos 3B + \cos 3C - 1 = 0$

$\Rightarrow \cos 3A + \cos 3B + \cos 3C + \cos 3\pi = 0$

$$\Rightarrow 2\cos\left(\frac{3A+3B}{2}\right)\cos\left(\frac{3A-3B}{2}\right) + 2\cos\left(\frac{3\pi+3C}{2}\right)\cos\left(\frac{3\pi-3C}{2}\right) = 0$$

$$\Rightarrow 2\cos\left(\frac{3\pi-3C}{2}\right)\left\{\cos\left(\frac{3A-3B}{2}\right) + \cos\left(\frac{3\pi+3C}{2}\right)\right\} = 0$$

$$\Rightarrow 2\cos\left(\frac{3\pi}{2}-\frac{3C}{2}\right)2\cos\left(\frac{3\pi+3C+3A-3B}{4}\right)\cdot\cos\left(\frac{3\pi+3C-3A+3B}{4}\right) = 0$$

$$\Rightarrow 2\cos\left(\frac{3\pi}{2}-\frac{3C}{2}\right)2\cos\left(\frac{3\pi}{2}-\frac{3B}{2}\right)\cdot\cos\left(\frac{3\pi}{2}-\frac{3A}{2}\right) = 0$$

$$\Rightarrow -4\sin\left(\frac{3A}{2}\right)\sin\left(\frac{3B}{2}\right)\sin\left(\frac{3C}{2}\right) = 0$$

$$\Rightarrow \sin\left(\frac{3A}{2}\right)\sin\left(\frac{3B}{2}\right)\sin\left(\frac{3C}{2}\right) = 0$$

$\therefore \dfrac{3A}{2} = \pi$ or $\dfrac{3B}{2} = \pi$ or $\dfrac{3C}{2} = \pi$

$\therefore A = \dfrac{2\pi}{3}$ or $B = \dfrac{2\pi}{3}$ or $C = \dfrac{2\pi}{3}$

2. (d) : Let $I = \sum_{r=0}^{10} \cos^3 \dfrac{\pi r}{3}$

$$= \sum_{r=0}^{10} \frac{1}{4}\left(\cos 3\frac{\pi r}{3} + 3\cos\frac{\pi r}{3}\right)$$

$$= \sum_{r=0}^{10} \frac{1}{4}\left(\cos \pi r + 3\cos\frac{\pi r}{3}\right) = \frac{1}{4}(I_1 + I_2)$$

where $I_1 = \sum_{r=0}^{10} \cos \pi r = 1 - 1 + 1 - 1 + ... - 1 + 1 = 1$

$$I_2 = 3\sum_{r=0}^{10} \cos\frac{\pi r}{3} = \frac{3\cos\left(\frac{10}{2}\frac{\pi}{3}\right)\sin\frac{11\pi}{6}}{\sin\frac{\pi}{6}} = -\frac{3}{2}$$

$$\therefore\ I = \frac{1}{4}\left(1 - \frac{3}{2}\right) = -\frac{1}{8}$$

3. (d) : We have, $\sin^3 10° + \sin^3 50° - \sin^3 70°$

$$= \frac{1}{4}\left[(3\sin 10° - \sin 30°) + (3\sin 50° - \sin 150°) - (3\sin 70° - \sin 210°)\right]$$

$$= \frac{1}{4}\left[3(\sin 10° + \sin 50° - \sin 70°) - \frac{3}{2}\right]$$

$$= \frac{1}{4}\left[3(\sin 10° - 2\cos 60° \cdot \sin 10°) - \frac{3}{2}\right] = -\frac{3}{8}$$

4. (a) : We have, $2\sin^2 x + \dfrac{2\sin x \cos x}{2} = n$

$\Rightarrow \sin 2x - 2\cos 2x = 2n - 2$

$\Rightarrow -\sqrt{5} \le 2n - 2 \le \sqrt{5}$

$\Rightarrow 1 - \dfrac{\sqrt{5}}{2} \le n \le 1 + \dfrac{\sqrt{5}}{2}$

5. (c) : Since α, β, γ are the roots of $x^3 + px + q = 0$

$\therefore\ \alpha + \beta + \gamma = 0$

Applying $C_1 \to C_1 + C_2 + C_3$, then we get

$$\begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} = 0$$

6. (c) : Given, $2z^2 + 2z + k = 0$

$\therefore\ z = \dfrac{-2 \pm \sqrt{4 - 8k}}{4}$

Since 'z' is a complex number

$\therefore\ 4 - 8k$ will be negative $\Rightarrow k > \dfrac{1}{2}$

Now, $(0, 0), \left(\dfrac{-1}{2}, \dfrac{\sqrt{2k-1}}{2}\right)\left(\dfrac{-1}{2}, \dfrac{-1}{2}\sqrt{2k-1}\right)$ are the vertices of the triangle.

Since triangle is equilateral

$$\therefore\quad \frac{1}{4}(2k-1)+\frac{1}{4}=(2k-1)$$

$$\Rightarrow\quad k = 2/3$$

7. (b) : Given, $\dfrac{a+b}{2}=2\sqrt{ab}$

$$\Rightarrow\quad a+b-4\sqrt{ab}=0$$

$$\Rightarrow\quad \frac{a}{b}+1-4\sqrt{\frac{a}{b}}=0 \text{ (dividing by } b \text{ on both sides)}$$

$$\Rightarrow\quad \left(\sqrt{\frac{a}{b}}\right)^2-4\sqrt{\frac{a}{b}}+1=0$$

Hence, $\sqrt{\dfrac{a}{b}}=\dfrac{4\pm 2\sqrt{3}}{2}=(2\pm\sqrt{3})$

$$\therefore\quad \frac{a}{b}=\frac{2+\sqrt{3}}{2-\sqrt{3}}$$

8. (c) : Let $t_r=\dfrac{{}^nC_{r-1}}{{}^nC_r+{}^nC_{r-1}}=\dfrac{{}^nC_{r-1}}{{}^{n+1}C_r}=\dfrac{{}^nC_{r-1}}{\frac{n+1}{r}\,{}^nC_{r-1}}$

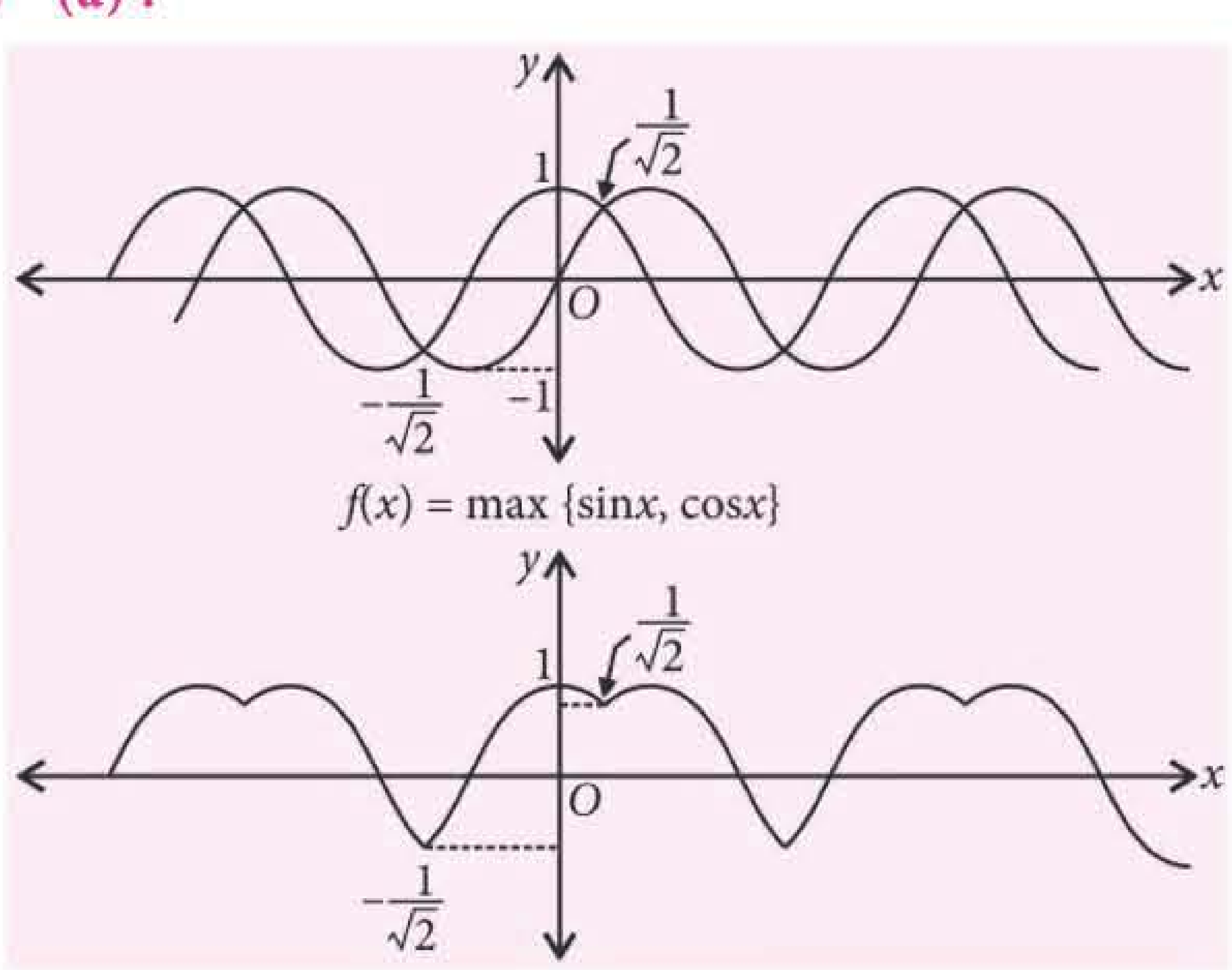
$$\therefore\quad t_r=\frac{r}{n+1}$$

Now, $S=\displaystyle\sum_{r=0}^{n}\{t_r\}^3=\sum_{r=0}^{n}\frac{r^3}{(n+1)^3}=\frac{1}{(n+1)^3}\sum_{r=0}^{n}r^3$

$$\Rightarrow\quad S=\frac{1}{(n+1)^3}\left\{\frac{n(n+1)}{2}\right\}^2=\frac{n^2}{4(n+1)}$$

Now, $S=\dfrac{25}{24}$ (given) which is only possible for $n = 5$.

9. (a) :



Required range $=\left[-\dfrac{1}{\sqrt{2}},1\right]$

10. (b) : Period of $f(x)$ is 2π, but $f(x)$ is not defined for $x\in(\pi/2, 3\pi/2)$. Hence it suffices to consider $x\in[-\pi/2,\pi/2]$. Further since $f(x)$ is even, we consider $x\in[0,\pi/2]$.

Now, $\sqrt{\cos(\sin x)}$ and $\sqrt{\sin(\cos x)}$ are decreasing functions for $x\in[0,\pi/2]$.

$$\Rightarrow\quad R_F=[f(\pi/2), f(0)]=\left[\sqrt{\cos 1}, 1+\sqrt{\sin 1}\right]$$

11. (b) : Let $u_k=\dfrac{k+2}{k(k+1)(k+3)}$

$$=\frac{(k+2)^2}{k(k+1)(k+2)(k+3)}$$

$$=\frac{k^2+4k+4}{k(k+1)(k+2)(k+3)}=\frac{k(k+1)+3k+4}{k(k+1)(k+2)(k+3)}$$

$$=\frac{1}{(k+2)(k+3)}+\frac{3}{(k+1)(k+2)(k+3)}+\frac{4}{k(k+1)(k+2)(k+3)}$$

$$=\left(\frac{1}{k+2}-\frac{1}{k+3}\right)-\frac{3}{2}\left[\frac{1}{(k+2)(k+3)}-\frac{1}{(k+1)(k+2)}\right]$$
$$-\frac{4}{3}\left[\frac{1}{(k+1)(k+2)(k+3)}-\frac{1}{k(k+1)(k+2)}\right]$$

Now, put $k = 1, 2, 3, ..., n$ and add.

Thus, $s_n = u_1 + u_2 + ... + u_n$

$$=\left(\frac{1}{3}-\frac{1}{n+3}\right)-\frac{3}{2}\left[\frac{1}{(n+2)(n+3)}-\frac{1}{2\cdot 3}\right]$$
$$-\frac{4}{3}\left[\frac{1}{(n+1)(n+2)(n+3)}-\frac{1}{1.2.3}\right]$$

Therefore $\displaystyle\lim_{n\to\infty}s_n=\frac{1}{3}+\frac{3}{12}+\frac{4}{18}=\frac{29}{36}$

12. (b) : Let $P=\dfrac{(n!)^{\frac{1}{n}}}{n}=\left(\dfrac{(n!)}{n^n}\right)^{\frac{1}{n}}$

$$\therefore\quad \log P=\frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$$

$$\Rightarrow\quad \log P=\int_0^1\log x\,dx \Rightarrow \log P=-1 \Rightarrow P=e^{-1}.$$

13. (c) : Differentiating given equation we get

$$e^{-x}\cdot f'(x)-e^{-x}\cdot f(x)=\sqrt{1+x^4} \qquad ...(i)$$

Since $(gof)(x) = x$ as 'g' is inverse of f.

$$\Rightarrow\quad g[f(x)]=x \Rightarrow g'[f(x)]f'(x)=1$$

$$\Rightarrow\quad g'[f(0)]=\frac{1}{f'(0)} \Rightarrow g'(2)=\frac{1}{f'(0)}$$

(Here $f(0) = 2$ obtained from given equation)

Put $x = 0$ in (i), we get $f'(0) = 3$.

$\therefore \quad g'(2) = \frac{1}{3}$.

14. (c) : When $y = 0$, $f(0) = 1$ and when $x = 0$, $f(-y) = \frac{1}{f(y)}$

Hence $f(x + y) = f(x)f(y)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f(x)\lim_{h\to 0}\frac{f(h)-1}{h}$$
$$= f(x).f'(0) = pf(x)$$

Put $x = 5$ in above equation we get, $f(5) = \frac{q}{p}$

15. (d) : The given function is

$$f(x) = \begin{cases} \tan^{-1}x & \text{if } |x|\le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x|>1 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} \frac{1}{2}(-x-1) & \text{if } x<-1 \\ \tan^{-1}x & \text{if } -1\le x\le 1 \\ \frac{1}{2}(x-1) & \text{if } x>1 \end{cases}$$

Clearly L.H.L at $(x = -1) = \lim_{h\to 0} f(-1-h) = 0$

R.H.L. at $(x = -1) = \lim_{h\to 0} f(-1+h)$

$$= \lim_{h\to 0}\tan^{-1}(-1+h) = -\pi/4$$

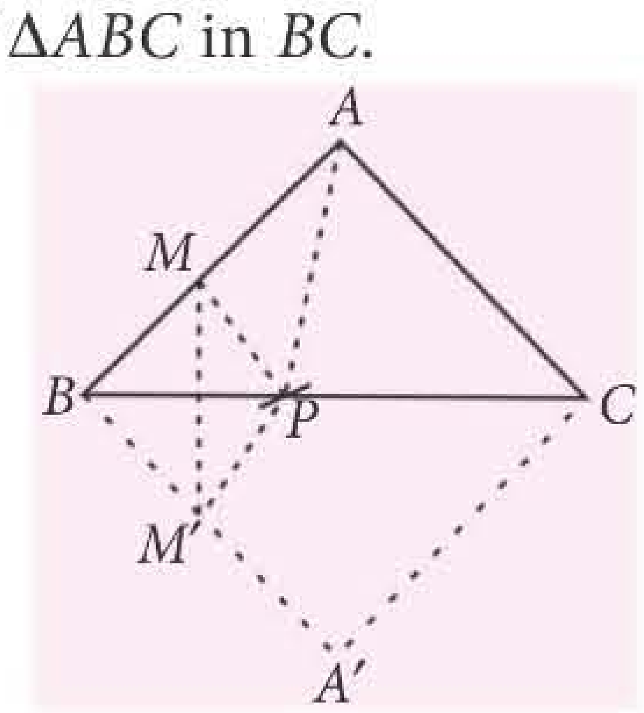
Since L.H.L $\neq$ R.H.L. at $x = -1$

$\therefore \quad f(x)$ is discontinuous at $x = -1$

Also we can prove in the same way that $f(x)$ is discontinuous at $x = 1$

$\therefore \quad f(x)$ can not be found for $x = \pm 1$ or domain of $f'(x) = R - \{-1, 1\}$.

16. (a) : Take the reflection of ΔABC in BC.



Now, $PM = PM'$

$PA + PM = PA + PM'$ it is minimum when M' PA lies in a line.

Now apply cosine rule in $\Delta ABM'$,

we get $AM' = 10\sqrt{7}$.

17. (a) : Coordinates of A and B are $(-3, 4)$ and $\left(-\frac{3}{5}, \frac{8}{5}\right)$.

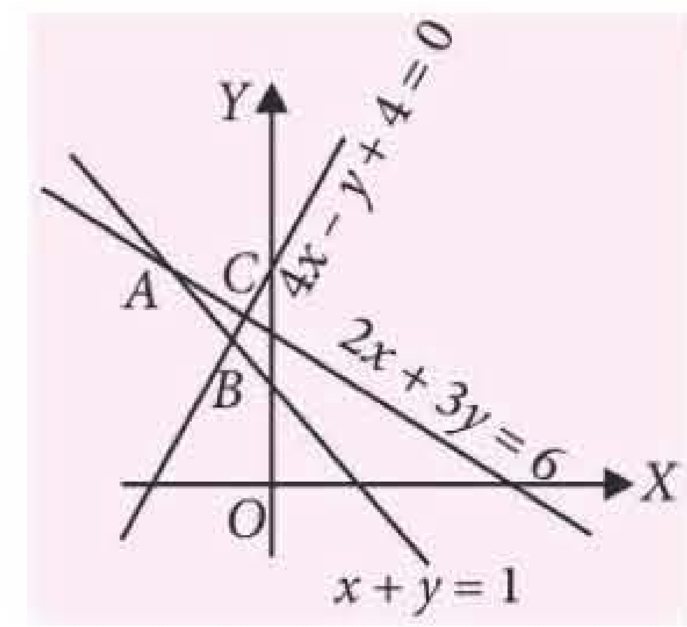
If orthocentre is $P(h, k)$, then

(slope of PA) $\times$ (slope of BC) $= -1$

$$\Rightarrow \quad \frac{k-4}{h+3}\times 4 = -1 \;\Rightarrow\; 4k - 16 = -h - 3$$

$\Rightarrow \quad h + 4k = 13$ ...(i)

And (slope of PB) $\times$ (slope of AC) $= -1$



$$\Rightarrow \quad \frac{k-\frac{8}{5}}{h+\frac{3}{5}}\times\left(-\frac{2}{3}\right) = -1$$

$$\Rightarrow \quad \frac{5k-8}{5h+3}\times\frac{2}{3} = 1$$

$\Rightarrow \quad 10k - 16 = 15h + 9$

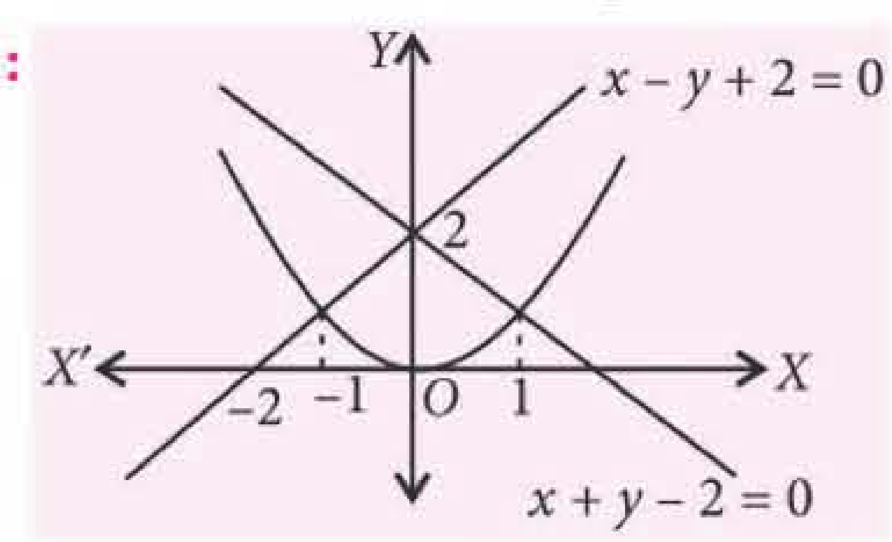
$\Rightarrow \quad 15h - 10k + 25 = 0$

$\Rightarrow \quad 3h - 2k + 5 = 0$ ...(ii)

Solving (i) and (ii), we get $h = \frac{3}{7}, k = \frac{22}{7}$

Hence, orthocentre lies in I quadrant.

18. (b) :

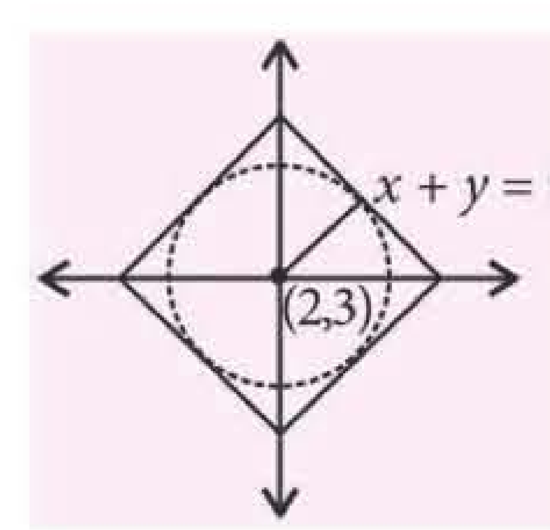


(a, a^2) lies on $y = x^2$

$a - a^2 + 2 = 0 \Rightarrow a = -1, 2$

$a + a^2 - 2 = 0 \Rightarrow a = 1, -2$

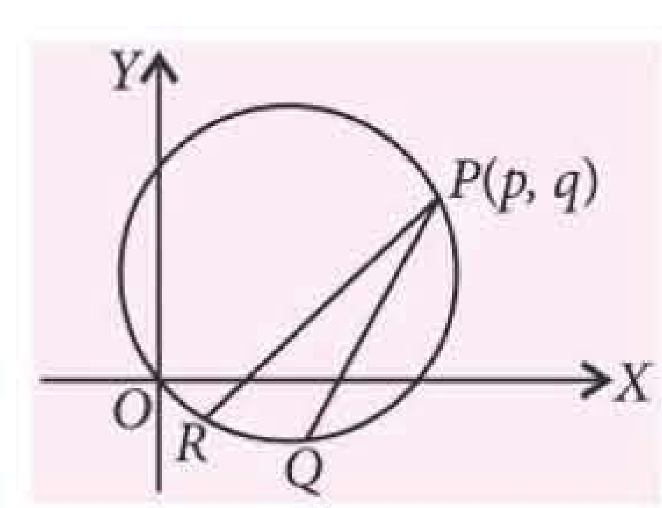
19. (d) : Perpendicular distance from centre to tangent = radius



$$r = \frac{|2+3-9|}{\sqrt{2}}$$
$$= \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

Equation of circle is $(x - 2)^2 + (y - 3)^2 = 8$

20. (d) : Let PQ be a chord of the given circle passing through $P(p, q)$ and $Q(x, y)$. Since PQ is bisected by the x-axis, the mid-point of PQ lies on the x-axis which gives $y = -q$



Now Q lies on the circle

$x^2 + y^2 - px - qy = 0$

So $x^2 + q^2 - px + q^2 = 0$

$\Rightarrow \quad x^2 - px + 2q^2 = 0$, ...(i)

Which gives two values of x and hence the coordinates of two points Q and R (say), so that the chords PQ and PR are bisected by x-axis.

If the chords PQ and PR are distinct, the roots of (i) are real distinct.

Hence, discriminant $= p^2 - 8q^2 > 0$

$\Rightarrow \quad p^2 > 8q^2$

21. (b) : Clearly $\alpha = -i$, where $i^2 = -1$

$$\text{Now, } \Delta(\alpha) = \alpha^n \frac{1}{\alpha^n}\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^3 \\ \frac{1}{\alpha} & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -i & i \\ i & 1 & 0 \end{vmatrix}$$

$$= 1(-i) + 1(i^2) + (1 + i^2) = -1 - i$$

So, principal argument of $\Delta(\alpha)$ is $-\frac{3\pi}{4}$

22. (b) : If $\Delta = \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$ then other determinant (say Δ^1) is the cofactor determinant

Since, $\Delta\Delta^1 = \Delta^3$(for 3rd order det)

$\because \quad \Delta = \lambda(\lambda^2 + a^2 + b^2 + c^2)$

$\therefore$ By comparing we get $\lambda = 1$.

23. (b) : By solving the equations of sides of the rhombus, the vertices are

$$\left(0, \frac{-n}{m}\right), \left(\frac{-n}{l}, 0\right)\left(0, \frac{n}{m}\right), \left(\frac{n}{l}, 0\right)$$

$$\therefore \text{ The area} = \frac{1}{2}\left(\frac{2n}{m}\right)\left(\frac{2n}{l}\right) = 2 \Rightarrow n^2 = lm$$

24. (a) : Third vertex 'A' lies on $x - y = 0$ and in III quadrant

Perpendicular distance from (0,0) to $x + y = 1$ is $\frac{1}{\sqrt{2}}$

$\therefore \quad AO = \sqrt{2} \Rightarrow A(-1, -1)$

If m is the slope of other side, $\tan 60° = \left|\frac{m+1}{1-m}\right|$

$\Rightarrow \quad m = 2 \pm \sqrt{3}$

25. (c) : $f(x) = x^2 + x + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{1}{2} \geq \frac{1}{2}$

Now, $g(f(x)) = (f(x))^2 + af(x) + 1$, for $g(f(x)) = 0$

$$a = -\left(f(x) + \frac{1}{f(x)}\right) \leq -2$$

$\therefore$ If $a > -2$, $g(f(x)) = 0$ has no solutions.

26. (c) : $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$ but < 1

$$\therefore \quad \left[\frac{n\sin\theta}{\theta}\right] = n - 1$$

Also $\left[\frac{n\tan\theta}{\theta}\right] = n \qquad \left[\because \frac{\tan\theta}{\theta} \to 1 \text{ as } \theta \to 0 \text{ but} > 1\right]$

27. (a) : $\sin^2 x - a \sin x + b = 0$ has only one solution in $(0, \pi)$.

$\Rightarrow$ $\sin x = 1$ gives one solution and $\sin x = \alpha$ gives other solution such that $\alpha > 1$ or $\alpha \leq 0$

$\Rightarrow$ $(\sin x - 1)(\sin x - \alpha) = 0$ is the same equation as $\sin^2 x - a \sin x + b = 0$

$\Rightarrow$ $1 + \alpha = a$ and $\alpha = b$

$\Rightarrow$ $1 + b = a$ and $b > 1$ or $b \leq 0$

$\Rightarrow$ $b \in (-\infty, 0] \cup (1, \infty)$ and $a \in (-\infty, 1] \cup (2, \infty)$

28. (c) : Let $A(x_1, y_1)$ be any point outside the parabola and $B(\alpha, \beta)$, $C(\alpha', \beta')$ be the points of contact of tangents from point A. Equation of chord BC is $yy_1 = 2a(x + x_1)$

Length of the perpendicular (AL) from A to BC

$$= \frac{y_1^2 - 4ax_1}{\sqrt{y_1^2 + 4a^2}}$$

Now, area of $\Delta ABC = \frac{1}{2} \times (AL \times BC)$

$$= \frac{(y_1^2 - 4ax_1)^{3/2}}{2a} \quad \left[\because BC = \frac{\{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)\}^{1/2}}{a}\right]$$

29. (b) : $x^n - 1 = (x - 1)(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_{n-1})$

$= x^n - x^{n-1}(1 + \alpha_1 + \ldots + \alpha_{n-1})$

$$+x^{n-2}\left(\sum_{1 \leq i < j \leq n-1} \alpha_i\alpha_j + \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}\right) + \ldots - 1 = 0$$

$$\Rightarrow \sum_{1 \leq i < j \leq n-1} \alpha_i\alpha_j + \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} = 0$$

$$\Rightarrow \sum_{1 \leq i < j \leq n-1} \alpha_i\alpha_j = 1$$

30. (b) : Sum $= \{C_0 + (C_0 + C_1 + C_2 + \ldots + C_{n-1})\}$

$+ \{(C_0 + C_1) + (C_0 + C_1 + \ldots + C_{n-2})\} + \{(C_0 + C_1 + C_2)$

$+ (C_0 + C_1 + \ldots + C_{n-3})\} + \ldots$ to $\left(\frac{n}{2}\right)$ terms

$$= (C_0 + C_1 + \ldots\ldots C_n) \times \frac{n}{2} = n \cdot 2^{n-1}$$

Challenging PROBLEMS ON JEE Main

1. Given that $x_1 + x_2 + x_3 = 0$, $y_1 + y_2 + y_3 = 0$ and $x_1y_1 + x_2y_2 + x_3y_3 = 0$. Then
$$\frac{x_1^2}{x_1^2+x_2^2+x_3^2}+\frac{y_1^2}{y_1^2+y_2^2+y_3^2}=$$
(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) 1 (d) $\frac{4}{3}$

2. $\sum_{r=1}^{n} r(r+1)(r+2)...(r+p) =$ (where n and p are positive integers)
(a) $\frac{n(n+1)(n+2)....(n+p+1)}{(p+2)}-(p+1)!$
(b) $\frac{n(n+1)(n+2)....(n+p+1)}{(n+p+2)}$
(c) $\frac{n(n+1)(n+2)....(n+p)}{(p+2)}$
(d) $\frac{n(n+1)(n+2)....(n+p)}{(n+p+2)}$

3. Let $z = (18 + 26i)$ and $z_0 = a + ib$ is the cube root of z having least positive argument then $a + b =$
(a) 1 (b) 2 (c) 3 (d) 4

4. Consider the two lines L_1, L_2 and a circle C
$L_1 : 2x + 3y + p - 3 = 0$, $L_2 : 2x + 3y + p + 3 = 0$
$C : x^2 + y^2 + 6x + 10y + 30 = 0$, $(p \in I)$
It is given that at least one of the lines L_1, L_2 is a chord of C then the probability that both are chords of C is
(a) $\frac{2}{7}$ (b) $\frac{3}{7}$ (c) $\frac{4}{7}$ (d) $\frac{5}{7}$

5. Let $f(x)=\log\left(\frac{x^2+e}{x^2+1}\right)$
and $g(x)=\sqrt{\sin f(x)}+\sqrt{\cos f(x)}$, then range of $g(x)$ is
(a) $(1, 2^{3/4}]$ (b) $(2^{1/2}, 2^{3/4}]$
(c) $(\pi^{1/2}, \pi^{3/4})$ (d) $(e^{1/2}, \pi^{1/2}]$

6. If the parabola $y = ax^2 + bx + c$ has vertex at (4, 2) and $a \in [1, 3]$ then maximum value of product (a b c) is
(a) 144 (b) 12 (c) -12 (d) -144

7. For $p \geq 2$, the equation
$\sqrt{2p+1-x^2}+\sqrt{3x+p+4}=\sqrt{x^2+9x+3p+9}$ has
(a) exactly 1 real root
(b) exactly 2 distinct real roots
(c) exactly 3 distinct real roots
(d) no real roots

8. If $f : R \to R$, $g : R \to R$ and $f(x) + f''(x) = -xg(x)f'(x)$ and $g(x) > 0$ $\forall$ $x \in R$ then $f^2(x) + (f'(x))^2$ has
(a) a minima at $x = 0$
(b) a maxima at $x = 0$
(c) a point of inflexion at $x = 0$
(d) data insufficient

9. All bases of logarithms in which a real positive number can be equal to its logarithm is/ are
(a) $(0, 1) \cup (1, e^{1/e}]$ (b) $(1, e)$
(c) $(1, e^{1/2})$ (d) $[e^{1/e}, e^e)$

10. Let $f(x)=\lim_{m\to 0}\frac{1}{m^4}\cdot\int_0^m \frac{(e^{x+t}-e^x)(\log(1+t))^2}{3+2t^3}dt$
then $f(\log 3) =$
(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{12}$

11. The non-negative real numbers $x_r (r = 1$ to $5)$ satisfy the following relations :
(1) $\sum_{r=1}^{5} r\cdot x_r = a$ (2) $\sum_{r=1}^{5} r^3 x_r = a^2$ and

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(3) $\sum_{r=1}^{5} r^5 x_r = a^3$, then number of possible values of a is/are

(a) 0 (b) 1 (c) 6 (d) 12

12. Let $\sum_{k=1}^{\infty} \cot^{-1}\left(\frac{k^2}{8}\right) = \frac{-\pi}{n}$, $n \in I$ then $n =$

(a) 1 (b) 2 (c) 4 (d) 8

13. The natural number n for which $2^8 + 2^{11} + 2^n$ is a perfect square is

(a) 11 (b) 12 (c) 14 (d) 15

14. If $f(x)$ is a differentiable function defined $\forall\, x \in R$ such that $(f(x))^3 = x - f(x)$ then $\int_0^{\sqrt{2}} f^{-1}(x)dx =$

(a) 1 (b) 2 (c) $\sqrt{2}$ (d) $2\sqrt{2}$

15. Let $f(x) = \begin{cases} e^{\{x^2\}} - 1, & x > 0 \\ 0, & x = 0 \\ \dfrac{\sin x - \tan x + \cos x - 1}{2x^2 + \tan x + \log(x+2)}, & x < 0 \end{cases}$

Lines L_1 and L_2 represent tangent and normal to curve $y = f(x)$ at $x = 0$. Consider the family of circles touching both lines L_1 and L_2. Then the ratio of the radii of two orthogonally circles of this family is

(a) $2+\sqrt{2}$ (b) $2+\sqrt{3}$ (c) $2-\sqrt{2}$ (d) $2-\sqrt{3}$

16. Suppose a and b are single digit positive integers chosen independently and at random. The probability that the point (a, b) lies above the parabola $y = ax^2 - bx$ is

(a) $\frac{17}{81}$ (b) $\frac{19}{81}$ (c) $\frac{21}{81}$ (d) $\frac{23}{81}$

17. Let $f(x) = ax^2 + bx + c$, $a, b, c \in I$. Let $f(1) = 0$, $f(7) \in (50, 60)$, $f(8) \in (70, 80)$ then $f(2) \in$

(a) $(-2, 0)$ (b) $(0, 10)$ (c) $(1, 12)$ (d) $(20, 30)$

18. In ΔABC, H is the orthocentre and $AH \cdot BH \cdot CH = 3$ and $AH^2 + BH^2 + CH^2 = 7$ then sum of the possible circumradius (R) of the ΔABC is

(a) $\frac{2}{5}$ (b) $\frac{5}{2}$ (c) $\frac{3}{2}$ (d) $\frac{2}{3}$

SOLUTIONS

1. (b) : Consider three vectors

$\vec{n}_1 = (x_1, x_2, x_3)$, $\vec{n}_2 = (y_1, y_2, y_3)$ and $\vec{n}_3 = (1,1,1)$.

From the given data,

$\vec{n}_1 \cdot \vec{n}_2 = 0, \vec{n}_2 \cdot \vec{n}_3 = 0$ and $\vec{n}_1 \cdot \vec{n}_3 = 0$

i.e., $\vec{n}_1, \vec{n}_2$ and $\vec{n}_3$ are mutually $\perp^r$ vectors.

Now, $\frac{x_1^2}{x_1^2 + x_2^2 + x_3^2}, \frac{y_1^2}{y_1^2 + y_2^2 + y_3^2}$ and $\frac{1}{3}$ are the squares of the projections of the vector $(1, 0, 0)$ onto the direction of $\vec{n}_1, \vec{n}_2, \vec{n}_3$ respectively and hence their sum = 1

i.e., $\frac{x_1^2}{x_1^2 + x_2^2 + x_3^2} + \frac{y_1^2}{y_1^2 + y_2^2 + y_3^2} + \frac{1}{3} = 1$

2. (a) : Since $r(r+1)\,.....\,(r+p) = (p+1)!\;{}^{r+p}C_{p+1}$

$= (p+1)!\;[{}^{r+p+1}C_{p+2} - {}^{r+p}C_{p+2}]$

and required sum $= (p+1)!\;[{}^{n+p+1}C_{p+2-1}]$

$= \frac{n(n+1).....(n+p+1)}{p+2} - (p+1)!$

3. (d) : $z = 18 + 26i = 10\sqrt{10}\,[\cos\theta + i\sin\theta]$

where $\tan\theta = \frac{13}{9} \Rightarrow \tan\left(\frac{\theta}{3}\right) = \frac{1}{3}$

and $z_0 = z^{1/3} = \sqrt{10}\,[\cos(\theta/3) + i\sin(\theta/3)]$

$z_0 = \sqrt{10}\left[\frac{3}{\sqrt{10}} + \frac{i}{\sqrt{10}}\right] = 3 + i$

4. (b) : For L_1 to be a chord of the circle, possible integral value of p are {17, 18, ..., 31}. Similarly, for L_2 to be a chord, possible integral value of p are {11, 12, ..., 25}. So, in total possible $p = 21$ and common values are 9.

Hence, probability $= \frac{9}{21} = \frac{3}{7}$

5. (a) : Notice that $\frac{x^2 + e}{x^2 + 1} \in [1, e]$

Hence, $f(x) \in (0, 1]$

So, $g(\alpha) = \sqrt{\sin\alpha} + \sqrt{\cos\alpha}$, $\alpha \in (0, 1]$

$g'(\alpha) = 0$ gives $\alpha = \pi/4$

So, $g(x) \in (1, 2^{3/4}]$

6. (d) : From given data, $\frac{-b}{2a} = 4$ and $\frac{-D}{4a} = 2$

So, $c = 2 + 16a$

and $E = abc = -16(a^2 + 8a^3)$

So, $\frac{dE}{da} = -16(2a + 24a^2) < 0$, for $a \in [1, 3]$

Hence, $E_{max.} = -16(1^2 + 8 \cdot 1^3) = -144$

7. (b) : Let $h = x^2 + x - p$, then given equation becomes

$\sqrt{(x+1)^2 - 2h} + \sqrt{(x+2)^2 - h} = \sqrt{(2x+3)^2 - 3h}$

Simplifying further, $h[2h - 2(x+2)^2 - (x+1)^2] = 0$

If $2h - 2(x+2)^2 - (x+1)^2 = 0$ then above square root equation is invalid. Hence, only $h = 0$ possible. So, now above equation becomes

$|x+1| + |x+2| = |2x+3| \Rightarrow x \notin (-2, -1)$

Hence, the number of real solutions of $h = x^2 + x - p = 0$ which are not in $(-2, -1)$ is zero if $p < -\frac{1}{4}$, one if $p = -\frac{1}{4}$ or $p \in (0, 2)$ and two otherwise. Hence, exactly 2 real roots for $p \in \left(\frac{-1}{4}, 0\right] \cup [2, \infty)$.

8. (b) : Let $h(x) = (f(x))^2 + (f'(x))^2$

So, $h'(x) = -2x\, g(x) \cdot (f'(x))^2$

9. (a) : We require $a \in R - \{1\}$ such that $x = \log_a x$

i.e. $f(x) = \frac{\log x}{x} = \log a$

$f'(x) = 0 \Rightarrow x = e$

So, max. $f(x) = f(e) = 1/e$. So, $a_{\max.} = e^{1/e}$

Hence, $a \in (0, 1) \cup (1, e^{1/e}]$

10. (c) : $$f(x) = e^x \cdot \lim_{m\to 0} \frac{\int_0^m \frac{(e^t-1)\cdot(\log(1+t))^2}{2t^3+3}dt}{m^4} \quad \left(\frac{0}{0}\right)$$

$$\Rightarrow f(x) = e^x \cdot \lim_{m\to 0} \frac{(e^m-1)\cdot(\log(1+m))^2}{(3+2m^3)\cdot 4m^3}$$

$$\Rightarrow f(x) = e^x \cdot \lim_{m\to 0} \frac{e^m-1}{m}\cdot\left(\frac{\log(1+m)}{m}\right)^2\cdot\frac{1}{4}\cdot\frac{1}{3+2m^3}$$

$$\Rightarrow f(x) = \frac{e^x}{12}, \text{ so, } f(\log 3) = \frac{3}{12} = \frac{1}{4}$$

11. (c) : Notice that

$$\sum_{r=1}^{5} r(a-r^2)^2 x_r = a^2\sum_{r=1}^{5} r\cdot x_r - 2a\sum_{r=1}^{5} r^3 x_r + \sum_{r=1}^{5} r^5 x_r$$

$= a^2(a) - 2a(a^2) + (a^3) = 0$

Since, L.H.S. terms are non-negative.

Hence, each term in L.H.S. is zero.

So, possible values of a are $\{0, 1, 4, 9, 16, 25\}$

12. (c) : $$T_k = \cot^{-1}\left(\frac{k^2}{8}\right)$$

$$= \tan^{-1}\left(\frac{8}{k^2}\right) = \tan^{-1}\left[\frac{2}{1+\frac{k^2}{4}-1}\right]$$

$$= \tan^{-1}\left[\frac{k}{2}+1\right] - \tan^{-1}\left(\frac{k}{2}-1\right)$$

13. (b) : $2^8 + 2^{11} + 2^n = 2^8(9 + 2^{n-8})$

Hence, $9 + 2^{n-8}$ should be a perfect square.

So, $9 + 2^{n-8} = k^2$ (say) *i.e.* $2^{n-8} = (k+3)(k-3)$

So, $(k+3)$ and $(k-3)$ are both powers of 2.

$\Rightarrow k = 5$ being the only possibility. Hence, $n = 12$

14. (b) : $(f(x))^3 + f(x) = x$. Hence, $f^{-1}(x) = x^3 + x$

15. (b) : $$\text{L.H.D.} = \lim_{h\to 0}\left[\frac{-\sin h + \tan h + \cos h - 1}{2h^2 - \tanh + \log(2-h)}\right]\times\frac{1}{h} = 0$$

and $$\text{R.H.D.} = \lim_{h\to 0}\frac{e^{h^2}-1-0}{h} = 0$$

Hence, $L_1 = y = 0$ and $L_2 = x = 0$.

16. (b) : If (a, b) lies above the curve then $b > y(a)$

$$i.e., b > a^3 - ba \Rightarrow b > \frac{a^3}{a+1}$$

The only possibilities are $a = 1, 2, 3$

For $a = 1$, $b = 1, 2, 3, ...9$

For $a = 2$, $b = 3, 4, ... 9$

For $a = 3$, $b = 7, 8, 9$. So, in total there are 19 points out of $9 \times 9 = 81$ points.

Hence, required probability $= \frac{19}{81}$

17. (a) : $f(1) = 0 \Rightarrow a + b + c = 0$

$f(7) \in (50, 60) \Rightarrow 50 < 49a + 7b + c < 60$

or $50 < 48a + 6b < 60$ *i.e.*, $8a + b \in \left(\frac{25}{3}, 10\right)$

i.e., $8a + b = 9$

Similarly, $9a + b = 11$

Hence, $a = 2, b = -7, c = 5$

i.e., $f(x) = 2x^2 - 7x + 5 \Rightarrow f(2) = 8 - 14 + 5 = -1$

18. (b) : If ΔABC is acute, then Cosine rule gives,

$AB^2 = AH^2 + BH^2 - 2AH \cdot BH\cos(\pi - C)$ ...(i)

and $AB = 2R\sin C$, $CH = 2R\cos C$

$\Rightarrow AB^2 + CH^2 = 4R^2$ ...(ii)

From (i) and (ii),

$$4R^2 = AH^2 + BH^2 + CH^2 + \frac{AH\cdot BH\cdot CH}{R}$$

Now, A.T.Q., $4R^2 = 7 + \frac{3}{R}$ *i.e.*, $4R^3 - 7R - 3 = 0$

i.e., $(R+1)(2R+1)(2R-3) = 0$

Since, $3 = AH \cdot BH \cdot CH < (2R)^3 \Rightarrow R = 1$

Similarly when ΔABC is obtuse, we have $R = \frac{3}{2}$

So, sum of possibilities of $R = \frac{3}{2} + 1 = \frac{5}{2}$

◆◆

OLYMPIAD CORNER

1. Two unequal circles of radii R and r touch externally, and P and Q are the points of contact of a common tangent to the circles, respectively. Find the volume of the frustum of a cone generated by rotating PQ about the line joining the centres of the circles.

2. Let $n \geq 2$ be a natural number. Show that there exists a constant $C = C(n)$ such that for all real $x_1, \ldots, x_n \geq 0$ we have $\sum_{k=1}^{n} \sqrt{x_k} \leq \sqrt{\prod_{k=1}^{n} (x_k + C)}$.

   Determine the minimum $C(n)$ for some values of n.

3. Find all real coefficients polynomials $p(x)$ satisfying $(x-1)^2 p(x) = (x-3)^2 p(x+2)$ for all x.

4. Prove that : $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$,

   where x, y, z are non-negative real numbers for which $x + y + z = 1$.

5. Find the value of the continued root:

$$\sqrt{4+27\sqrt{4+29\sqrt{4+31\sqrt{4+33\sqrt{\ldots}}}}}$$

6. A hexagon is inscribed in a circle with radius r. Two of its sides have length 1, two have length 2 and the last two have length 3. Prove that r is a root of the equation $2r^3 - 7r - 3 = 0$.

7. Let $k \geq 2$ be an integer. The sequence (x_n) is defined by $x_0 = x_1 = 1$ and $x_{n+1} = \dfrac{x_n^k + 1}{x_{n-1}}$ for $n \geq 1$.

   (a) Prove that for each positive integer $k \geq 2$ the sequence (x_n) is a sequence of integers.

   (b) If $k = 2$, show that $x_{n+1} = 3x_n - x_{n-1}$ for $n \geq 1$.

8. If for every positive integer n, $f(n)$ is defined as

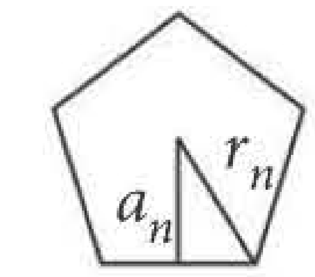
$$f(n) = \begin{cases} 1 & \text{for } n = 1 \\ \dfrac{n}{f(n-1)} & \text{for } n \geq 2 \end{cases}$$

   then prove that : $\sqrt{1992} < f(1992) < \frac{4}{3}\sqrt{1992}$.

9. We consider regular n-gons with a fixed circumference 4. We call the distance from the centre of such a n-gon to a vertex r_n and the distance from the centre to an edge a_n.

   (a) Determine a_4, r_4, a_8, r_8

   (b) Give an appropriate interpretation for a_2 and r_2.



   (c) Prove: $a_{2n} = \frac{1}{2}(a_n + r_n)$ and $r_{2n} = \sqrt{a_{2n} r_n}$.

   (d) Let $u_0, u_1, u_2, u_3, \ldots$ be defined as follows :

   $u_0 = 0$, $u_1 = 1$, $u_n = \frac{1}{2}\left(u_{n-2} + u_{n-1}\right)$ for even n

   and $u_n = \sqrt{u_{n-2}\, u_{n-1}}$ for odd n.

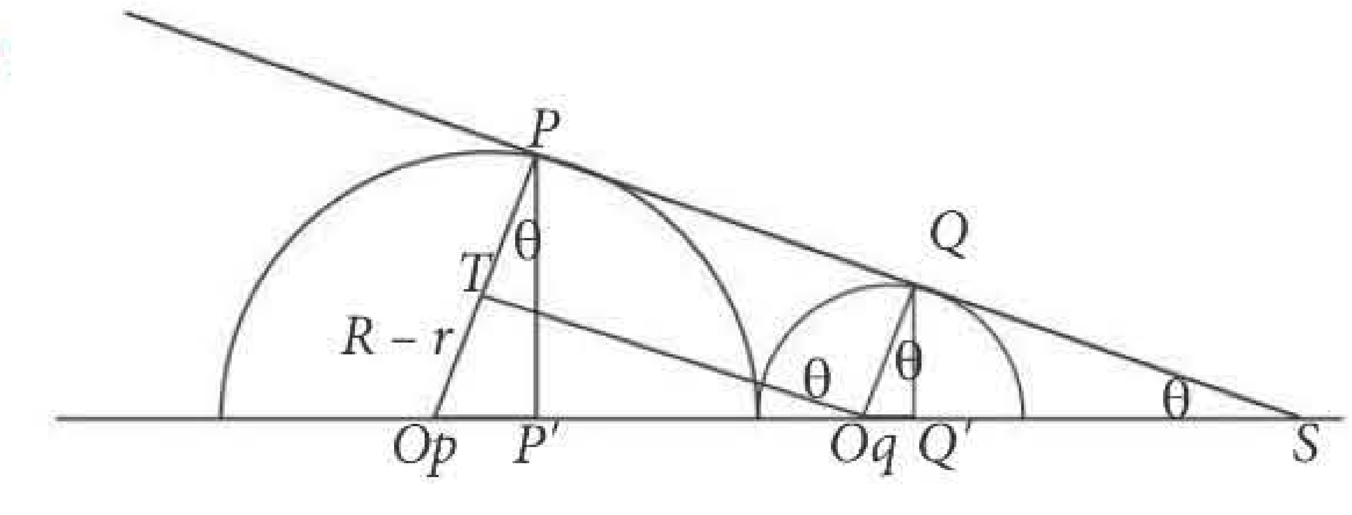
   Determine: $\lim_{n\to\infty} u_n$

10. There are real numbers a, b, c such that $a \geq b \geq c > 0$.

    Prove that $\dfrac{a^2 - b^2}{c} + \dfrac{c^2 - b^2}{a} + \dfrac{a^2 - c^2}{b} \geq 3a - 4b + c$.

SOLUTIONS

1.



First, we note that triangle $\Delta TOpOq$ has a right angle

angle at T, with $OpOq = R + r$ and $OpT = R - r$. Hence $OqT = 2\sqrt{Rr}$.

Because of parallel and perpendicular lines, all of angles $\angle PSOp$, $\angle TOqOp$, $\angle OpPP'$ and $\angle OqQQ'$ are equal. We denote the common value by θ. From triangle $\Delta OpOqT$, we note that

$$\sin\theta = \frac{R-r}{R+r} \text{ and } \cos\theta = \frac{2\sqrt{Rr}}{R+r}$$

Using various right triangles, we obtain:

$$OpP' = R\sin\theta = \frac{R(R-r)}{R+r},\ PP' = R\cos\theta = \frac{2R\sqrt{Rr}}{R+r}$$

$$P'S = PP'\cot\theta = \frac{2R\sqrt{Rr}}{R+r}\frac{2\sqrt{Rr}}{R-r} = \frac{4rR^2}{R^2-r^2},$$

$$QqQ' = r\sin\theta = \frac{r(R-r)}{R+r}$$

$$QQ' = r\cos\theta = \frac{2r\sqrt{Rr}}{R+r}$$

$$Q'S = QQ'\cot\theta = \frac{2r\sqrt{Rr}}{R+r}\frac{2\sqrt{Rr}}{R-r} = \frac{4r^2R}{R^2-r^2}$$

Hence, $V = \frac{\pi}{3}\left[(PP')^2P'S - (QQ')^2Q'S\right]$

$$= \frac{\pi}{3}\left[\frac{4R^3r}{(R+r)^2}\frac{4rR^2}{R^2-r^2} - \frac{4Rr^3}{(R+r)^2}\frac{4r^2R}{R^2-r^2}\right]$$

$$= \frac{16\pi R^2r^2(R^3-r^3)}{3(R+r)^2(R^2-r^2)} = \frac{16\pi R^2r^2(R^2+Rr+r^2)}{3(R+r)^3}$$

Which is the required volume.

2. We show that the inequality is valid for an aggregate of values of C of which the least is

$$C(n) = \frac{n-1}{\sqrt[n-1]{n^{n-2}}}, n \geq 2.$$

Let us first do the easier task of proving the existence of C's which make the inequality valid. Of course this part will be redundant as soon as we improve the technique to find the least C.

Setting $x_i = y_i^2$ where $y_i \geq 0$ ($i = 1,\ldots,n$), we are to show, equivalently, that for some C we have

$$\left(\sum_{i=1}^{n} y_i\right)^2 \leq \prod_{i=1}^{n} y_i^2 + C \qquad \text{...(i)}$$

Treating the right hand side of (1) as a polynomial in C, we observe that all coefficients are non-negative and that the coefficient of C^{n-1} is $\sum y_i^2$.

Thus, $\prod_{i=1}^{n} y_i^2 + C \geq \left(\sum_{i=1}^{n} y_i^2\right)C^{n-1}$

But by the Cauchy–Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} y_i\right)^2 \leq n\left(\sum_{i=1}^{n} y_i^2\right),$$

So inequality (i) will be valid if we choose $C = n^1/(n-1)$ or larger. This completes the easier task.

It turns out that $n^{1/(n-1)}$ is only a slight overestimate of the minimum C, which we now seek. for any C for which (i) is valid, set $w_i = y_i\sqrt{n-1}/\sqrt{C}$, so that (i) becomes

$$\left(\sum_{i=1}^{n} w_i\right)^2 \leq \frac{C^{n-1}}{(n-1)^{n-1}}\prod_{i=1}^{n}(w_i^2+n-1)$$

or equivalently

$$\left(\sum_{i=1}^{n} w_i\right)^2 \leq \frac{C^{n-1}n^n}{(n-1)^{n-1}}\prod_{i=1}^{n}\left(\frac{w_i^2-1}{n}+1\right) \qquad \text{...(ii)}$$

To find the minimum C we shall first show that the following inequality is valid:

$$\left(\sum_{i=1}^{n} w_i\right)^2 \leq n^2\prod_{i=1}^{n}\left(\frac{w_i^2-1}{n}+1\right) \qquad \text{...(iii)}$$

we shall use the weierstrass inequality

$$\prod_{i=1}^{m}(1+a_i) \geq 1 + \sum_{i=1}^{m} a_i$$

Which holds if all $a_i \leq 0$ or if $-1 < a_i < 0$ for all i. Without loss of generality let $w_1,\ldots w_t \geq 1$ and $0 \leq w_{t+1},\ldots, w_n < 1$, where $t \in \{0, 1,\ldots,n\}$. Then

$$\prod_{i=1}^{n}\left(\frac{w_i^2-1}{n}+1\right) = \prod_{i=1}^{n}\left(\frac{w_i^2-1}{n}+1\right).\ \prod_{i=t+1}^{n}\left(\frac{w_i^2-1}{n}+1\right)$$

$$\geq \left(1+\sum_{i=1}^{t}\frac{w_i^2-1}{n}\right)\left(1+\sum_{i=t+1}^{n}\frac{w_i^2-1}{n}\right)$$

$$= \frac{1}{n^2}\left(n-t+\sum_{i=1}^{t} w_i^2\right)\left(t+\sum_{i=t+1}^{n} w_i^2\right)$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{t} w_i^2 + \sum_{i=t+1}^{n} 1^2\right)\left(\sum_{i=1}^{t} 1^2 + \sum_{i=t+1}^{n} w_i^2\right) \geq \frac{1}{n^2}\left(\sum_{i=1}^{n} w_i\right)$$

(the last inequality by the Cauchy-Schwarz inequality), which proves (iii). Note that equality occurs for $w_1 = \ldots w_n = 1$. We conclude that (ii) is valid for any C with $C^{n-1}n^n/(n-1)^{n-1} \geq n^2$, *i.e.*, with

$$C \geq \frac{n-1}{\sqrt[n-1]{n^{n-2}}}, n \geq 2.$$

The minimum value $C(n)$ We seek is then as stated at the beginning, since for

$$x_i = y_i^2 = c\left(\frac{w_i}{\sqrt{n-1}}\right)^2 = \frac{C\,.\,1}{n-1}$$

the original inequality reduces to equality.

Remark : The above shows $C(2) = 1$,

$C(3) = \frac{2}{\sqrt{3}} \approx 1.1547$, $C(4) = \frac{3}{\sqrt[3]{4^2}} \approx 1.1905$,

$C(5) = \frac{4}{\sqrt[4]{5^3}} \approx 1.1963$, and generally $C(n) \approx n^{1/(n-1)}$ which approaches 1 in the limit.

3. We consider polynomials $p(x)$ with coefficients in a field F of arbitrary characteristic and find as follows:

(i) If char $(F) = 0$, (in particular, if $F = R$), then $p(x) = a(x-3)^2$, where a is any scalar (possibly 0) in F;

(ii) If char $(F) = 2$, then every $p(x)$ satisfies the equation (clear);

(iii) If char $(F) = 2$ an odd prime, l, then there are infinitely many solutions, including all $p(x) = a(x-3)^2 (x^{l^v} - x + c)$ with $a, c \in F$, and $v = 0, 1, 2, \ldots$ (Note that $p(x)$ has the form $a(x-3)^2$ if $v = 0$.

To prove this, observe that if char $(F) \neq 2$, then $x - 1$ and $x - 3$ are coprime, whence $p(x) = (x-3)^2 q(x)$ in $F[x]$.

Thus our equation becomes

$(x^{l^v} - 1)^2 (x-3)^2 q(x) = (x-3)^2 (x-1)^2 q(x+2) \quad (*)$

whence $q(x) = q(x+2)$, as polynomials; that is, elements of $F[x]$.

Now if char $(F) = 0$, then $(*)$ has only constant solution. (The most elementary proof of this: without loss of generally, $q(x) = x^n + ax^{n-1} + \ldots$. Then $q(x+2) - q(x) = 2nx^{n-1} + \ldots$, and this is non-zero if $n \geq 1$. Another proof: $(*)$ implies that $q(x)$ is periodic, which forces equations $q(x) = c$ to have infinitely many roots x, a contradiction).

This establishes the assertion (i).

Re: assertion (iii). Let char $(F) = l$ and

$q(x) = x^{l^v} - x + c$.

Then for $x = 0, 1, \ldots, l-1$, (that is for each element of the prime field), we have $q(x) = c$ and so $q(x) = q(x+1) = q(x+2) = \ldots$, yielding polynomials of degree greater than or equal to l which satisfy. $(*)$. This establishes the assertion (iii).

4. In this problem, we will prove that for $x, y, z \leq 0$,

$$0 \leq (yz + zx + xy)(x + y + z) - 2xyz \leq \frac{7}{27}(x + y + z)^3.$$

This is the homogeneous version of the original inequality. The expression in the middle expands to $\sum x^2 y + xyz$, which is clearly non-negative. We focus on the right inequality, which becomes $\sum x^2 y + xyz \leq \frac{7}{27} \sum x^3 + \frac{7}{9} \sum x^2 y + \frac{14}{9} \sum xyz$, which implies

$6 \sum x^2 y \leq 7 \sum x^3 + 15xyz$.

A property of homogeneous polynomials, and an alternate definition, is the following: $p(x_1, x_2, \ldots, x_n)$ is homogeneous of degree k if

$p(\lambda x_1, \lambda x_2, \ldots, x_n) = \lambda^k p(x^1, x^2, \ldots, x_n)$ for all $\lambda \in R$.

Going back to the original problem,

$p(x, y, z) = (yz + zx + xy)(x + y + z) - 2xyz$.

If $x + y + z = 0$, then all three variables must be 0, and the inequality follows. Otherwise, we can set $\lambda = \frac{1}{x+y+z}$, and then

$$0 \leq p\left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right) \leq \frac{7}{27}$$

5. More generally, for any positive integer n, we claim that $\sqrt{4 + n\sqrt{4 + (n+2)\sqrt{4 + (n+4)\sqrt{\ldots}}}} = n+2$, where the left side is defined as the limit of

$$F(n,m) = \sqrt{4 + n\sqrt{4 + (n+2)\sqrt{4 + (n+4)\sqrt{\ldots\sqrt{4 + m\sqrt{4}}}}}}$$

as $m \to \infty$ (where m is an integer and $(m - n)$ is even).

If $g(n, m) = F(n, m) - (n+2)$, we have

$F(n, m)^2 - (n+2)^2 = (4 + nF(n+2, m)) - (4 + n(n+4))$

$= n(F(n+2, m) - (n+4))$,

$$g(n,m) = \frac{n}{F(n,m) + n + 2} g(n+2, m).$$

Clearly $F(n, m) > 2$,

So, $|g(n, m)| < \frac{n}{n+4} |g(n+2, m)|$.

By iterating this, we obtain

$$|g(n, m)| < \frac{n(n+2)}{m(m+2)} |g(m,m)| < \frac{n(n+2)}{m}.$$

Therefore $g(n,m) \to 0$ as $m \to \infty$

Let $S_n = \sqrt{4 + (2n-1)\sqrt{4 + (2n+1)\sqrt{4 + (2n+3)\sqrt{\ldots}}}}$

S_n satisfies the recurrence relation

$S_n = \sqrt{4 + (2n-1)S_{n+1}}$ if and only if

$(S_n - 2)(S_n + 2) = (S_n - 1) S_n + 1$.

By inspection, this admits $S_n = 2n + 1$ as a solution. We only have to prove that $S_1 = 3$ to make this induction complete. Let

$$T_n = \sqrt{4+\sqrt{4+3\sqrt{\ldots(2n-3)\sqrt{4+2n-1\sqrt{(2n+3)}}}}}$$

and $U_n = \sqrt{4+\sqrt{4+3\sqrt{\ldots(2n-3)\sqrt{4+(2n-1)(2n+3)}}}} = 3$

Clearly $T_n \le U_n$ and the latter is identically equal to 3. Therefore, using the fact that $B \ge A > 0$ implies that $\sqrt{(4+A)/(4+B)} \ge \sqrt{A/B}$,

$$1 \le \frac{T_n}{3} = \frac{T_n}{U_n} = \frac{\sqrt{4+\sqrt{\ldots+(2n-1)\sqrt{2n+3}}}}{\sqrt{4+\sqrt{\ldots(2n-1)\sqrt{2n+3}}}}$$

$$\ge \frac{\sqrt{\sqrt{\ldots+(2n-1)\sqrt{2n+3}}}}{\sqrt{\sqrt{\ldots+(2n-1)\sqrt{2n+3}}}} \ge \ldots \ge 2^{n+1}\sqrt{\frac{1}{2n+3}}$$

$$\frac{1}{(2n+3)^{1/2^{n+1}}} \to 1$$

as $n \to \infty$ [for example, by rewriting as exp {– In $(2n + 3)/2^{n+1}$} and using L' Hopital's rule]. This proves that $S_1 = \lim_{n\to\infty} T_n = 3$. The required expression is precisely S_{14} and hence its value is 29.

6. Equal chords subtend equal angles at the centre of a circle; if each of sides of length i subtends an angle α_i (i = 1, 2, 3) at the centre of the given circle, then $2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 360°$,

where, $\frac{\alpha_1}{2} + \frac{\alpha_2}{2} = 90° - \frac{\alpha_3}{2}$,

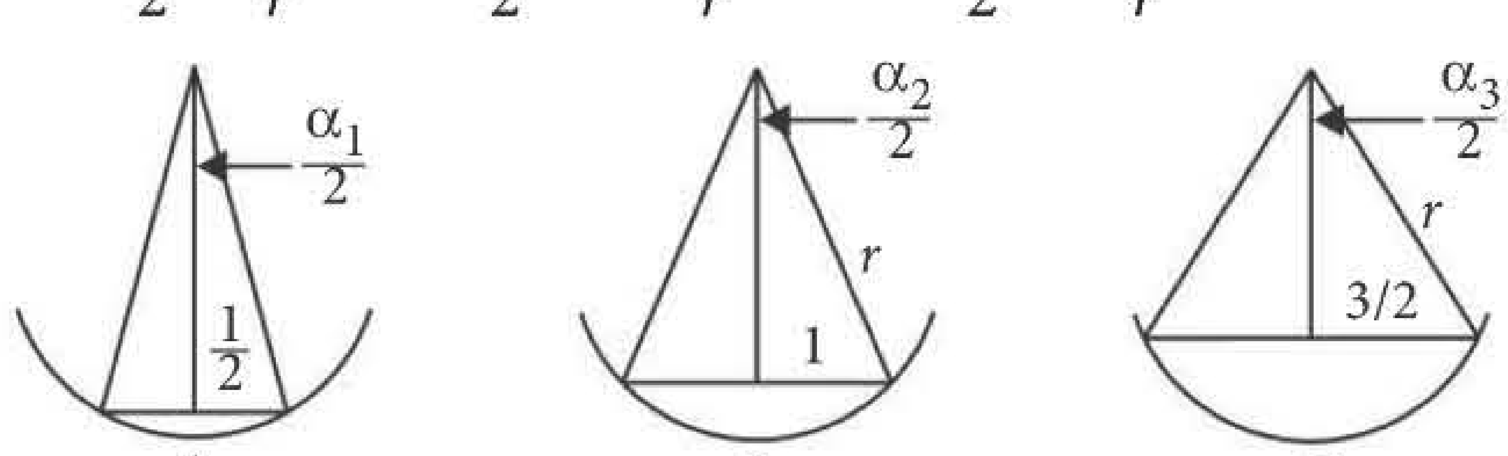
and $\cos\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right) = \cos\left(90° - \frac{\alpha_3}{2}\right) = \sin\frac{\alpha_3}{2}$,

Next we apply the addition formula for the cosine:

$$\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2} = \sin\frac{\alpha_3}{2}, \qquad \ldots(\text{i})$$

where, $\sin\frac{\alpha_1}{2} = \frac{1/2}{r}$ $\cos\frac{\alpha_1}{2} = \frac{\sqrt{4r^2-1}}{2r}$;

$\sin\frac{\alpha_2}{2} = \frac{1}{r}$, $\cos\frac{\alpha_2}{2} = \frac{\sqrt{r^2-1}}{r}$; $\sin\frac{\alpha_3}{2} = \frac{3/2}{r}$.



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We substitute these expressions into (i) and obtain, after multiplying both sides by $2r^2$,

$$\sqrt{4r^2-1}.\sqrt{r^2-1} - 1 = 3r.$$

Now write it in the form $\sqrt{(4r^2-1)(r^2-1)} = 3r+1$, and sqaure, obtaining $(4r^2 - 1)(r^2 - 1) = 9r^2 + 6r + 1$, which is equivalent to $r(2r^3 - 7r - 3) = 0$.

Since $r \ne 0$, we have $2r^3 - 7r - 3 = 0$, which was to be shown.

7. (a) We immediately get $x_2 = 2$ and $x_3 = 2^k + 1$. Now we use mathematical induction for the proof. Assume that $x_0, x_1, \ldots, x_n$ are all natural numbers. We must show that $x_{n+1} \in N$. First we note that since $x_{n-2} \cdot x_n = x_{n-1}^k + 1$ it follows that x_{n-2} and x_{n-1} are relatively prime. Using $x_n = (x_{n-1}^k + 1)/x_{n-2}$ we infer that

$$x_{n+1} = \frac{x_n^k + 1}{x_{n-1}} = \frac{(x_{n-1}^k + 1)^k + x_{n-2}^k}{x_{n-2}^k x_{n-1}}.$$

Thus obviously x_{n-2}^k divides $N = (x_{n-1}^k + 1)^k + x_{n-2}^k$ since x_n is a natural number. Furthermore, modulo x_{n-1} we have:

$N \equiv 1 + x^k_{n-2} = x_{n-3} \cdot x_{n-1} \equiv 0$.

That is, x_{n-1} also divides N and we are done.

(b) Now, $x_{n+1} = \frac{x_n^2+1}{x_{n-1}} \Leftrightarrow x_{n-1} \cdot x_{n+1} - x_n^2 = 1.$

That is, the sequence $\{y_n\} = \{x_{n-1} \cdot x_{n+1} - x_n^2\}$ is constant. Setting $y_{n+1} = y_n$ we have

$x_n \cdot x_{n+2} - x_{n+1}^2 = x_{n-1} \cdot x_{n+1} - x_n^2$

$\Leftrightarrow x_n(x_n + x_{n+2}) = x_{n+1}(x_{n-1} + x_{n+1})$

$\Leftrightarrow \frac{x_n + x_{n+2}}{x_{n+1}} = \frac{x_{n-1} + x_{n+1}}{x_n}.$

That is, the sequence $\{z_n\} = \{(x_{n-1} + x_{n+1})/x_n\}$ is constant. From $z_1 = 3$ we get $(x_{n-1} + x_{n+1})/x_n = 3$; that is, $x_{n+1} = 3x_n - x_{n-1}$ for all $n \ge 1$, as claimed.

8. In general, $\sqrt{n+1} < f(n) < \frac{4}{3}\sqrt{n}$ ...(i)

for all even $n \ge 6$. In particular, for $n = 1992$, we would get $\sqrt{1993} < f(1992) < \frac{4}{3}\sqrt{1992}$.

First note that $f(n) = \frac{n}{f(n-1)} = \frac{n}{n-1} f(n-2)$ for all $n \ge 3$. If $N = 2k$ where $k \ge 2$, then multiplying $f(2q) = \frac{2q}{2q-1} f(2q-2)$ for q = 2, 3, ..., k, we get

$$f(2k) = \frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2k}{2k-1}.f(2)$$

$$= \left(\frac{2}{1}\right)\cdot\left(\frac{4}{3}\right)\cdot\left(\frac{6}{5}\right)\cdots\left(\frac{2k}{2k-1}\right) > \left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\left(\frac{7}{6}\right)\cdots\left(\frac{2k+1}{2k}\right).$$

Hence, $(f(2k))^2 > \dfrac{2.4.6...2k}{1.3.5...(2k-1)} \cdot \dfrac{3.5.7...(2k+1)}{2.4.6...2k} = 2k+1,$

$\therefore \quad f(n) = f(2k) > \sqrt{2k+1} = \sqrt{n+1}.$ ...(ii)

On the other hand, for $k \geq 3$ we have

$$2(2k) = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)...\left(\frac{2k-2}{2k-1}\right).2k$$

$$< \left(\frac{2}{3}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)...\left(\frac{2k-2}{2k}\right)2k.$$

Hence, $(f(2k))^2 < \left(\dfrac{2}{3}\right)^2 \cdot \dfrac{4.6...(2k-2)}{5.7...(2k-1)} \cdot \dfrac{5.7...(2k-1)}{6.8...2k}.(2k)^2$

$$= \left(\frac{2}{3}\right)^2 .4.2k$$

from which it follows that

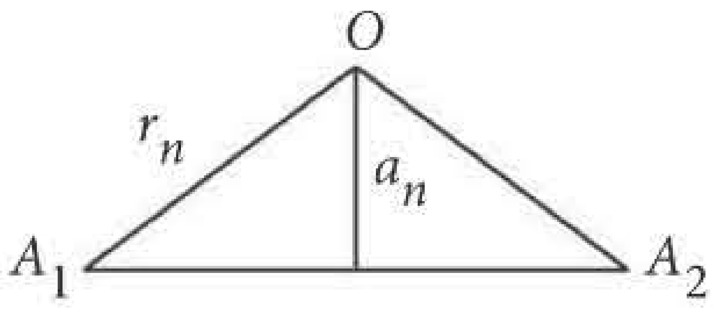
$\therefore \quad f(n) = f(2k) < \dfrac{4}{3}\sqrt{2k} = \dfrac{4}{3}\sqrt{n}.$ ...(iii)

The result follows from (ii) and (iii).

Note: Using similar arguments, upper and lower bounds for $f(n)$ when n is odd can also be easily derived. In fact, if we set $P = \dfrac{1.3.5...(2k-1)}{2.4.6..2k}$ (usually denoted by $\dfrac{2k-1!}{(2k)!}$, then various upper and lower bounds for P abound in the literature; for example, it is known that $\dfrac{1}{2}\sqrt{\dfrac{5}{4k+1}} \leq P \leq \dfrac{1}{2}\sqrt{\dfrac{3}{2k+1}}$

and $\dfrac{1}{\sqrt{\left(n+\dfrac{1}{2}\right)\pi}} < P \leq \dfrac{1}{\sqrt{n\pi}}.$

9. Let O be the centre of the regular n-gon. Let A_1A_2 denote one side of the regular n-gon



Then, we have $\angle A_1OA_2 = \dfrac{2\neq}{n}$, $\angle OA_1A_2 = \angle OA_2A_1$

$= \dfrac{\pi}{2} - \dfrac{\pi}{n}$. Thus, $|\overline{A_1A_2}| = \sqrt{r_n^2 + r_n^2 - 2r_n^2 \cos\dfrac{2\pi}{n}}$

$$= \sqrt{2r_n^2\left(1-\cos\frac{2\pi}{n}\right)} = \sqrt{4r_n^2 \sin^2\frac{\pi}{n}} = 2r_n \sin\frac{\pi}{n}.$$

The circumference of the regular n-gon is

$2nr_n \sin\dfrac{\pi}{n} = 4$ whence $r_n = \dfrac{2}{n\sin\dfrac{\pi}{n}},$

$$a_n = r_n \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = r_n \cos\frac{\pi}{n} = \frac{2}{n}\cot\frac{\pi}{n}.$$

In particular

$$r_4 = \frac{1}{2}\frac{1}{\sin\frac{\pi}{4}} = \frac{\sqrt{2}}{2}, \quad a_4 = \frac{2}{4}\cot\frac{\pi}{4} = \frac{1}{2},$$

$$r_8 = \frac{2}{8\sin\frac{\pi}{8}} = \frac{1}{4\sin\frac{\pi}{8}}.$$

Now, $\cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} = 1 - 2\sin^2\dfrac{\pi}{8}$

$\therefore \quad \sin\dfrac{\pi}{8} = \dfrac{1}{2}\sqrt{2-\sqrt{2}},$

So, $r_8 = \dfrac{1}{4}\dfrac{2}{\sqrt{2-\sqrt{2}}} = \dfrac{1}{2} \cdot \dfrac{1}{\sqrt{2-\sqrt{2}}},$

and $a_8 = r_8 \cos\dfrac{\pi}{8} = \dfrac{1}{4}\sqrt{\dfrac{2+\sqrt{2}}{2-\sqrt{2}}} = \dfrac{1}{4}\dfrac{1}{2-\sqrt{2}}\sqrt{2},$

since $\cos\dfrac{\pi}{4} = 2\cos^2\dfrac{\pi}{8} - 1.$

For (b), $r_2 = 1$, $a_2 = 0$ as the 2-gon is a straight line with O lying at the middle of A_1 and A_2.

For (c), we have

$$a_n + r_n = r_n\left(1+\cos\frac{\pi}{n}\right) = 2r_n \cos^2\frac{\pi}{2n}$$

$$= \frac{4}{n\sin\frac{\pi}{n}}\cos^2\frac{\pi}{2n} = \frac{4}{2n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}\cos^2\frac{\pi}{2n} = \frac{2}{n}\cot\frac{\pi}{2n}.$$

Thus $\dfrac{1}{2}(a_n + r_n) = \dfrac{1}{n}\cot\left(\dfrac{\pi}{2n}\right) = a_{2n}$, and

$$a_{2n}r_n = \frac{1}{n}\frac{\cos\frac{\pi}{2n}}{n\sin\frac{\pi}{2n}} \cdot \frac{2}{n\sin\frac{\pi}{n}} = \frac{1}{n^2} \cdot \frac{\cos\frac{\pi}{2n}}{\sin^2\frac{\pi}{2n}\cos\frac{\pi}{2n}} = \frac{1}{n^2\sin^2\frac{\pi}{2n}},$$

so $\sqrt{a_{2n}r_n} = \dfrac{1}{n\sin\dfrac{\pi}{2n}} = r_{2n}.$

For (d), note $u_0 = 0$, $u_1 = 1$, and $u_2 = \dfrac{1}{2}$. For $n \geq 2$ we have that u_n is either the arithmetic or geometric mean of u_{n-1} and u_{n-2} and in either case lies between them. It is also easy to show by induction that u_0, u_2, u_4, ... form an increasing sequence, and u_1, u_3, u_5, ... form a decreasing sequence with $u_{2l} \leq u_{2s+1}$ for all l,

$s \geq 0$. Let $\lim\limits_{k\to\infty} u_{2k} = P$ and $\lim\limits_{k\to\infty} u_{2k+1} = I$. Then $P \leq I$. We also have from $u_{2n} = \frac{1}{2}(u_{2n-1} + u_{2n-2})$ that $P = \frac{1}{2}(I + P)$ so that $I = P$ and $\lim\limits_{n\to\infty} u_n$ exists. Let $\lim\limits_{n\to\infty} u_n = L$. With $a_2 = 0$ and $r_2 = 1$, let $\overline{u}_{2k} = a_{2^{k+1}}$ and $\overline{u}_{2k+1} = r_{2^{k+1}}$, for $k = 0, 1, 2, \ldots$ From (c), $\overline{u}_0 = a_{2^1} = a_2 = 0$ and $\overline{u}_1 = r_{2^1} = r_2 = 1$. Also for $n = 2k + 2$, $\overline{u}_{2k+2}$

$= a_{2^{k+1+1}} = a_{2.2^{k+1}} = \frac{1}{2}\left(a_{2^{k+1}} + b_{2^{k+1}}\right) = \frac{1}{2}(\overline{u}_{2k} + \overline{u}_{2k+1})$;

that is $\overline{u}_n = \frac{1}{2}(\overline{u}_{n-2} + \overline{u}_{n-1})$ and for $n = 2k + 3$

$\overline{u}_{2k+3} = \overline{u}_{2(k+1)+1} = r_{2^{k+1+1}} = r_{2\left(2^{k+1}\right)}$

$= \sqrt{a_{2(2^{k+1})} . r_{2^{k+1}}} = \sqrt{a_{2^{k+1+1}} . r_{2^{k+1}}} = \sqrt{\overline{u}_{2(k+1)} \cdot \overline{u}_{2k+1}}$

so $\overline{u}_n = \sqrt{\overline{u}_{n-1} \cdot \overline{u}_{n-2}}$. Thus u_n and $\overline{u}_n$ satisfy the same recurrence and it follows that $L = \lim\limits_{k\to\infty} a_{2^{k+1}} = \lim\limits_{k\to\infty} r_{2^{k+1}}$. Now, from the solution to (c),

$$r_n = \frac{2}{n\sin\frac{\pi}{n}} = \frac{2}{\pi}\frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}}$$

So, $\lim\limits_{n\to\infty} r_n = \frac{2}{\pi}$ since $\frac{\neq}{n} \to 0$. Therefore $\lim\limits_{n\to\infty} u_n = \frac{2}{\pi}$.

10. From $a \geq b \geq c > 0$, we have

$$\frac{a+b}{c} \geq 2. \quad 0 < \frac{b+c}{a} \leq 2 \quad \text{and} \quad \frac{a+c}{b} \geq 1.$$

Now, $\frac{a^2 - b^2}{c} \geq 2(a - b)$, because $a \geq b$;

$\frac{c^2 - b^2}{a} \geq 2(c - b)$, because $c \leq b$

and $\frac{a^2 - c^2}{b} \geq a - c$, because $a \geq c$

After addition of these inequalities, we have

$$\frac{a^2 - b^2}{c} + \frac{c^2 - b^2}{a} + \frac{a^2 - c^2}{b} \geq 2(a - b) + 2(c - b) + (a - c),$$

that is, $\frac{a^2 - b^2}{c} + \frac{c^2 - b^2}{a} + \frac{a^2 - c^2}{b} \geq 3a - 4b + c$.

The equality holds if and only if $a = b = c > 0$.

QUANTITATIVE APTITUDE TEST

Useful for Bank PO, Specialist Officers & Clerical Cadre, BCA, MAT, CSAT, CDS and other such examinations

1. A three digit number has digits a, b, c from the left to right with $a > c$. If the digits are reversed & the number thus formed is subtracted from the original number, the unit digit is 3. What are the other two digits from left to right?
(a) 6 & 9 (b) 9 & 6 (c) 6 & 3 (d) 9 & 3

2. A person has five iron rods with lengths 16, 24, 48, 72, 104 cm each. He wants to convert into pieces of equal length from each of five rods. The least number of total pieces, if there is no wastage of material is
(a) 14 (b) 8 (c) 33 (d) 44

3. In a university, one third boys and half of the girls participate in the camp. Out of the total participants of 300 students, 100 are boys then find the total number of students in the university.
(a) 600 (b) 800 (c) 500 (d) 700

4. If $4x + 5y = 83$ and $\frac{3x}{2y} = \frac{21}{22}$, then $y - x$ is equal to
(a) 3 (b) 4 (c) 7 (d) 2

5. The square root of $\dfrac{\left(1\frac{1}{2}\right)^4 - \left(1\frac{1}{8}\right)^4}{\left(1\frac{1}{2}\right)^2 - \left(1\frac{1}{8}\right)^2}$ is
(a) $1\frac{7}{8}$ (b) $\frac{7}{8}$ (c) $2\frac{7}{8}$ (d) $1\frac{7}{64}$

6. The cube root of $135\sqrt{3} - 87\sqrt{6}$ equals
(a) $3\sqrt{2} + \sqrt{6}$ (b) $3\sqrt{3} - \sqrt{6}$
(c) $2\sqrt{3} - \sqrt{6}$ (d) None of these

7. Mahavir purchase 1000 articles at the rate ₹ 5 each and sold 850 articles at the rate ₹ 7 each and rest articles at the rate ₹ 3.50 each. Find the average profit per article sold.
(a) ₹ 1.50 (b) ₹ 2.47 (c) ₹ 1.47 (d) ₹ 1.75

8. The sum of three numbers is 264. If the first number be doubled the second and third number be one-third of the first, then the second number is
(a) 48 (b) 54 (c) 72 (d) 84

9. 10 years before, the ratio of ages of A and B was 13 : 17. 17 years from now, the ratio of their ages will be 10 : 11. The present age of B (in years) is
(a) 23 (b) 27 (c) 40 (d) 44

10. The average age of family of five members is 24 years. If the present age of youngest member is 8 years. What was the average age of the family at the time of birth of youngest member?
(a) 16 years (b) 18 years
(c) 20 years (d) 21 years

11. The tank full of petrol in Arun's motor cycle lasts for 10 days. If he starts using 25% more everyday, in how many days will the tank full of petrol last?
(a) 6 (b) 7 (c) 8 (d) 5

12. Rajni purchased a mobile phone and a refrigerator for ₹ 12000 and ₹ 10000 respectively, she sold the refrigerator at a loss of 12% and mobile phone at a profit of 8%. What is the overall loss/profit in whole transaction?
(a) loss of ₹ 280 (b) loss of ₹ 240
(c) profit of ₹ 2060 (d) loss of ₹ 640

13. A merchant earn a profit of 20% by selling a basket containing 80 apples, which cost ₹ 240, but he gave $\frac{1}{4}$ of it to his friend at cost price and sells the remaining apple. In order to earn the same profit, at what price he sells each apple?
(a) ₹ 3.00 (b) ₹ 3.60 (c) ₹ 3.80 (d) ₹ 4.80

14. Two vessels are full of milk with milk-water ratio 1 : 3 and 3 : 5 respectively. If both are mixed in the ratio 3 : 2, the ratio of milk and water in the new mixture is
(a) 4 : 15 (b) 3 : 7 (c) 3 : 10 (d) 6 : 7

15. Anil is an active and Vimal is a sleeping partner in a business. Anil invests ₹ 12000 and Vimal invests ₹ 20000. Anil received 10% profit for managing and the rest being divided in proportion to their capitals. Out of the total profit ₹ 9000, the money received by Anil is
(a) ₹ 4800 (b) ₹ 3937.50
(c) ₹ 4600 (d) ₹ 4500

16. *A* and *B* started a business with ₹ 20000 and ₹ 35000 respectively. They agreed to share the profit in the ratio of their capital. *C* joins the partnership with the condition that *A*, *B* and *C* will share profit equally and pays ₹ 220000 as premium for this, to be shared between *A* and *B*. This is to be divided between *A* and *B* in the ratio of
(a) 10 : 9 (b) 1 : 10 (c) 10 : 1 (d) 9 : 10

17. A daily wages worker appointed on a contract is paid ₹ 350 every day. He attends work and ₹ 125 is deducted from his salary as a fine every day remains absent. If in a month of 31 working days, he earned ₹ 8475, then for how many days he is absent?
(a) 5 (b) 6 (c) 7 (d) 8

18. 2000 soldiers in a fort had enough food for 20 days. But some soldiers were transferred to another fort and food lasted for 25 days. How many soldiers were transferred?
(a) 525 (b) 500 (c) 450 (d) 400

19. Nishtha can do a piece of work in 25 days and Tina can finish it in 20 days. They work together for 5 days then Nishtha quit herself. In how many days will Tina finish the remaining work?
(a) 20 (b) 18
(c) 15 (d) None of these

20. Through an inlet, a tank takes 8 hours to get filled up. Due to a leak in the bottom it takes 2 hours more to get it filled completely, if the tank is full, how much time will the leak take to empty it?
(a) 20 hours (b) 25 hours
(c) 40 hours (d) 30 hours

21. Two pipes *P* and *Q* can fill a cistern in 12 and 15 minutes respectively. If both are opened together and at the end of 3 minutes the tap *P* is closed. How much longer will the cistern take to fill?
(a) $8\frac{1}{4}$ minutes (b) $8\frac{1}{2}$ minutes
(c) $8\frac{3}{4}$ minutes (d) $9\frac{1}{4}$ minutes

22. A train of length 150 m takes 10 s to cross another train 100 m long coming from opposite direction. If speed of first train is 30 km/h. What is the speed of second train?
(a) 60 km/h (b) 55 km/h
(c) 50 km/h (d) 45 km/h

23. A motor boat takes 2h to travel a distance of 9 km down the current and it takes 6 h to travel the same distance against the current. What is the speed of current (stream or water flow)?
(a) 3 km/h (b) 2 km/h
(c) 1.5 km/h (d) 2.5 km/h

24. In two types of stainless steel, the ratio of chromium and steel are in ratio 2 : 11 and 5 : 21. In what proportion should the two typed be mixed, so that the ratio of chromium to steel in the mixture becomes 7 : 32?
(a) 2 : 3 (b) 3 : 4 (c) 1 : 2 (d) 1 : 3

25. How many kilograms of the tea powder costing ₹ 34 per kg be mixed with 33 kg of tea powder costing ₹ 42 per kg, such that the mixture when sold at ₹ 46 per kg gives a profit of 15%?
(a) 18 kg (b) 15 kg (c) 14 kg (d) 11 kg

26. An amount is invested in a bank at compound rate of interest. The total amount including interest after first year and third year is ₹ 1200 and ₹ 1587 respectively. What is the rate of interest?
(a) 10% (b) 12% (c) 15% (d) 20%

27. *ABC* is a triangle right angled at *A*, *AB* = 6 cm, *AC* = 8 cm. Semi circles are drawn (out side the triangle) on the sides *AB*, *AC* & *BC* as diameters which enclose the area *x*, *y* and *z* respectively. Then $z + y$ equals
(a) 17π cm^2 (b) $9\pi/2$ cm^2
(c) 20.5π cm^2 (d) $25\pi/2$ cm^2

28. A cylindrical box of radius 4 cm consisting of 6 solid spherical balls, each of radius same as the radius of cylindrical box. If the upper most ball touches upper cover of the box, then volume of the empty space in the box is
(a) 16π cm^3 (b) 64π cm^3
(c) 256π cm^3 (d) None of these

29. The clock *A* and *B* began to strike 12 together. Clock *A* strikes its strokes in 33 seconds and the clock *B* strikes its strokes in 22 seconds. What is the difference of interval between the 5th stroke of clock *A* and the 7th stroke of the clock *B*?
(a) 0 sec (b) 2 secs (c) 3 secs (d) 4 secs

30. If $\sin x + \cos x = p$ & $\sin^3 x + \cos^3 x = q$, then what is $p^3 - 3p$ equals?
(a) 0 (b) $-2q$ (c) $2q$ (d) $4q$

SOLUTIONS

1. **(a) :** Let original number $= 100a + 10b + c$ ...(i)
After interchanging the digits number $= a + 10b + 100c$ ...(ii)
Now, subtracting (ii) from (i), we have
Required number $= 99(a - c)$
As, unit digit in $99(a-c)$ is 3, so $a - c = 7$
$\therefore$ 3 digit number $= 99 \times 7 = 693$

2. **(c)** 3. **(d)** 4. **(b)** 5. **(a)**

6. **(b):** $135\sqrt{3} - 87\sqrt{6} = 3\sqrt{3}(45 - 29\sqrt{2})$
$\therefore (135\sqrt{3} - 87\sqrt{6})^{1/3} = \sqrt{3}(45 - 29\sqrt{2})^{1/3}$
Again, let $(45 - 29\sqrt{2})^{1/3} = x - \sqrt{y}$ ...(i)
$\therefore (45 + 29\sqrt{2})^{1/3} = x + \sqrt{y}$ ...(ii)
Now multiplying (i) and (ii), we get
$(2025 - 1682)^{1/3} = x^2 - y \Rightarrow x^2 - y = (343)^{1/3}$
$\Rightarrow y = x^2 - 7$
Again cubing (i), we get
$45 - 29\sqrt{2} = x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y}$ ...(iii)
Equating rational parts of (iii)
$x^3 + 3xy = 45 \Rightarrow 4x^3 - 21x - 45 = 0$
$\Rightarrow x = 3$(by using remainder theorem)
$\therefore y = 9 - 7 = 2$
Hence, $(135\sqrt{3} - 87\sqrt{6})^{1/3} = \sqrt{3}(3 - \sqrt{2})$

7. **(c)**

8. **(c) :** Let the second number be x.
Then first number is $2x$ and third number is $\frac{2x}{3}$.
According to problem, $2x + x + \frac{2x}{3} = 264$
$\Rightarrow 11x = 3 \times 264 \Rightarrow x = 72$

9. **(b):** In the problem $x = 13k$, $y = 17k$
$\therefore$ Present age of A is $13k + 10$ and B is $17k + 10$
$\therefore$ According to problem, we have
$$\frac{13k + 10 + 17}{17k + 10 + 17} = \frac{10}{11} \Rightarrow 27k = 27 \Rightarrow k = 1$$
$\therefore$ Present age of $B = 17k + 10 = 17(1) + 10 = 27$ years

10. **(c)**

11. **(c) :** Let the quantity of petrol used everyday be x.
$\therefore$ Quantity of petrol used for 10 days $= 10x$
= total petrol available
As the petrol used 25% more everyday
$\therefore$ Quantity of petrol used everyday $= x + \frac{x}{4} = \frac{5}{4}x$
$\therefore$ Required number of days
$$= \frac{\text{Total petrol available}}{\text{Petrol used everyday}} = \frac{10x}{\frac{5x}{4}} = \frac{40}{5} = 8$$

12. **(b)** 13. **(c)**

14. **(c) :** The ratio of milk and water in new mixture be x.
Now, according to problem, and using law of mixture we have
$$\frac{\frac{3}{8} - x}{x - \frac{1}{4}} = \frac{3}{2} \Rightarrow \frac{3 - 8x}{4x - 1} = \frac{3}{1} \Rightarrow x = \frac{3}{10}$$
$\therefore$ Required ratio is 3 : 10

15. **(b):** Anil's share for managing the business
$= 10\%$ of ₹ 9000 = ₹ 900
The profit which is distributed = ₹ 8100
Now ratio of investment = 12000 : 20000 = 3 : 5
$\therefore$ Anil's share $= 8100 \times \frac{3}{8} = 3037.50$
$\therefore$ Total amount received by Anil = ₹ 3937.50

16. **(c)** 17. **(a)** 18. **(d)** 19. **(d)**

20. **(c) :** Let the leak takes x hours to empty the tank.
Now, part filled by inlet in 1 hour = 1/8
Part filled by inlet and leak (outlet) together in 1 hour $= \frac{1}{8+2} = \frac{1}{10}$
Now, according to problem $\frac{1}{x} = \frac{1}{8} - \frac{1}{10} \Rightarrow x = 40$

21. **(a)**

22. **(a) :** Speed of I[st] train = 30 km/h $= \frac{25}{3}$ m/s
Total length of both the trains = 250 m
Let speed of II[nd] train be x m/s
Total time = 10 s
$$= \frac{\text{Total distance}}{\text{Speed of I}^{\text{st}}\text{ train} + \text{Speed of II}^{\text{nd}}\text{ train}}$$
$$\Rightarrow 10 = \frac{250}{25/3 + x} \Rightarrow \frac{250 \times 3}{3x + 25} = 10 \Rightarrow x = \frac{50}{3}$$
$\therefore$ Speed of II[nd] train $= \frac{50}{3} \times \frac{18}{5}$ km/h = 60 km/h

23. **(c)** 24. **(c)** 25. **(d)** 26. **(c)**
27. **(c)** 28. **(c)** 29. **(a)** 30. **(b)**